ADVANCED DIFFERENTIAL GEOMETRY II - SPIN GEOMETRY

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CONTENTS

0. MOTIVATION

Let T be a free particle in $\mathbb{R}^3$ with spin $1/2$. We want to study its motion in special relativity. If we denote its (relativistic) mass, Energy and momentum by m, E and p, respectively, then we have the relation

$$E = \sqrt{c^2 p^2 + m^2 c^4}\,, \tag{0.1}$$

where c denotes the speed of light.

Now we want to additionally study T quantum mechanically which means we have to describe T by a wave function $\psi = \psi_T : \mathbb{R} \times \mathbb{R}^3 \ni (t, x) \mapsto \psi(t, x) \in \mathbb{C}$. Here, the associated function $(t, x) \mapsto |\psi(t, x)|^2 \in \mathbb{R}$ is the density of the probability law that the particle T can be found at x at time t. The energy and momentum are no longer scalars associated with T but become unbounded operators acting on appropriate Hilbert spaces of wave functions,

$$\begin{aligned} E\psi &= ih\frac{\partial \psi}{\partial t}\,, \\ p\psi &= -ih\,\mathrm{grad}\,\psi\,. \end{aligned} \tag{0.2}$$

If one wants to combine the relativistic equation (0.1) with the quantum mechanical description (0.2), one concludes that wave functions must (formally) satisfy the equation

$$ih\frac{\partial \psi}{\partial t} = \sqrt{c^2 h^2 \Delta + m^2 c^4}\,\psi\,,$$

where Δ denotes the Laplacian $\Delta = -\sum_{i=1}^{3} \partial^2/\partial x_i^2$. *We thus face the problem of finding the square root of a second order differential operator*. Setting all constants to 1 (as mathematicians like to do), we specifically want to find the square root $D = \sqrt{\Delta}$ of the Laplacian. There are many ways in which this can be done, e.g., via the functional

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calculus, but for many reasons it is desirable that D be a differential operator itself. This means of course that D must be of first order. We take the ansatz

$$D = \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x_i}\,.$$

The requirement $D^2 = \Delta$ holds if and only if

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \qquad \text{and} \qquad \gamma_i\gamma_j + \gamma_j\gamma_i = 0\,, \qquad \text{for } i \neq j\,.$$

These equations do not posses a solution in $\mathbb{C}$. They do, however, if we allow the γ_i to be elements of some algebra. The smallest algebra that contains elements satisfying these relations is the one of complex 2×2-matrices. Specifically, the matrices

$$\gamma_1 = \begin{pmatrix} i & 0 \\ o & -i \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

do satisfy above equations. Now D becomes an operator acting on $\mathbb{C}^2$-valued functions, i.e. elements of $C^1(\mathbb{R}^3, \mathbb{C}^2)$, and the equation $D^2 = \Delta$ has to be understood component-wise.

This discussion was specific to $\mathbb{R}^3$. In the following lecture, we will learn how to define the Dirac operator D on (almost) any Riemannian manifold and study its basic properties.

1. Basics

1.1. Lie groups.

Definition 1.1. *A Lie group is a C^∞-manifold G which is also a group with the property that*

$$G \times G \ni (a,b) \mapsto a \cdot b \in G$$
$$G \ni a \mapsto a^{-1} \in G$$

are smooth.

Example 1.2.
(i) $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$, $(\mathbb{C}\backslash\{0\} = \mathbb{C}^*, \cdot)$.
(ii) $(S^1 = \{\mathrm{e}^{\mathrm{i}t} \,|\, t \in \mathbb{R}\} \subseteq \mathbb{C}^*, \cdot)$.
(iii) *If G, H are Lie groups, then $G \times H$ is a Lie group with the product manifold and product group structure.*
(iv) $(\mathrm{Gl}(n;\mathbb{C}), \cdot)$ *since* $\mathrm{Gl}(n;\mathbb{C})$ *is an open subset of* $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ *and matrix multiplication and inversion are polynomials in the entries of matrices, hence smooth. More generally,* $(\mathrm{Gl}(n;\mathbb{H}), \cdot)$*, where* $\mathbb{H}$ *is the field of quaternions.*
(v) *Any subgroup / submanifold of any Lie group G which also happens to be a submanifold / subgroup. For* $G = \mathrm{Gl}(n;\mathbb{C})$ *or* $G = \mathrm{Gl}(n;\mathbb{H})$ *the most prominent examples are:* $\mathrm{Gl}(n;\mathbb{R})$, $\mathrm{Sl}(n;\mathbb{C})$, $\mathrm{Sl}(n;R)$, $\mathrm{U}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n)$*. The groups* $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *are special cases of the following more general construction: Let* $\mathbb{K}$ *be either* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$ *and* $s : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ *a bi-/sesquilinear, nondegenerate (skew-)symmetric / (skew-)hermitian form. Then* $\mathrm{O}(s) := \{A \in M(n,n;\mathbb{K}) \,|\, s(AX, AY) = s(X,Y) \text{ for all } X, Y \in \mathbb{K}^n\}$ *is a Lie group.*
(vi) *The Heisenberg group*

$$H_{2n+1} := \left\{ \gamma(x,y,z) := \begin{pmatrix} 1 & x^t & z \\ 0 & E_n & y \\ 0 & 0 & 1 \end{pmatrix} \,|\, x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \subseteq \mathrm{Gl}(n;\mathbb{R})\,.$$

As a manifold, H_{2n+1} *is diffeomorphic to* $\mathbb{R}^{2n+1}$*. Group product and inversion are given by*

$$\gamma(x,y,z) \cdot \gamma(u,v,w) = \gamma(x+u, y+v, z+w+\langle x, v\rangle_{\text{eucl}})\,,$$
$$\gamma(x,y,z)^{-1} = \gamma(-x, -y, -z + \langle x, y\rangle_{\text{eucl}})\,.$$

Definition 1.3. *(i) For $a \in G$ the map $L_a : G \ni b \mapsto a \cdot b \in G$ is called left-translation by a . L_a is a diffeomorphism with inverse $L_a^{-1} = L_{a^{-1}}$. Analogously, $R_a : G \ni b \mapsto b \cdot a \in G$ right-translation by a .*
(ii) A vector field $X \in \mathcal{V}(G)$ is called left-invariant $:\Leftrightarrow$

$$\boxed{X \circ L_a = \mathrm{d}(L_a) \circ X \quad \forall a \in G\,,}$$

i.e., $X_{a \cdot b} = \mathrm{d}(L_a)_b X_b$ for all $a, b \in G$. In other words, X is L_a-related to itself for all $a \in G$.

Remark 1.4. *The space of left-invariant vector fields on G is canonically identified with T_eG, the tangent space to G at the identity:*

$$T_eG \ni X \mapsto (\text{vector field } \widetilde{X} \text{ given by } \widetilde{X}_a := \mathrm{d}(L_a)_e X_e)$$
$$T_eG \ni Y_e \leftarrow\!\shortmid Y \in \{\text{left-invariant vector fields on } G\}$$

These two maps are vector space isomorphisms and inverses of each other.

Lemma 1.5. *If X, Y are left-invariant vector fields on G, then $[X, Y]$ is again a left-invariant vector field.*

Proof. Let $a \in G$. Then X is L_a-related to itself, and so is Y. Hence, $[X, Y]$ is L_a-related to itself. □

Corollary and Definition 1.6. *(i) A Lie algebra over $\mathbb{R}$ is a real vector space V together with a bilinear map $[\cdot,\cdot] : V \times V \to V$ which is alternating and satisfies the Jacobi identity, i.e., $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in V$.*
(ii) The vector space $\mathfrak{g}$ of left-invariant vector fields on G is by Lemma 1.5 a Lie algebra over $\mathbb{R}$.

Remark 1.7. *The tangent space T_eG is canonically identified with $\mathfrak{g}$ by Remark 1.4. This means that T_eG inherits a Lie algebra structure from $\mathfrak{g}$!*

Explicitly: If $X, Y \in T_eG$, then $[X, Y] := [\text{left-inv. ext. } \widetilde{X} \text{ of } X, \text{left-inv. ext. } \widetilde{Y} \text{ of } Y]_e$.

One often encouters the notation $\mathfrak{g} = T_eG$, which should always be understood in the above sense.

Lemma 1.8. *Let X be a left-invariant vector field on G and Φ^t_X its flow. If $\Phi^t_X(e)$ is defined for all $t \in (-\varepsilon, \varepsilon)$, then so is $\Phi^t_X(a)$, and we have*

$$\boxed{\Phi^t_X(a) = a \cdot \Phi^t_X(e)\,.}$$

Proof. We need to check that $t \mapsto a \cdot \Phi^t_X(e)$ is an integral curve of X starting in a. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(a \cdot \Phi^t_X(e)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(L_a \Phi^t_X(e)\right) = d(L_a)_{\Phi^t_X(e)} \frac{\mathrm{d}}{\mathrm{d}t}\Phi^t_X(e) \\ &= \mathrm{d}(L_a)_{\Phi^t_X(e)} X_{\Phi^t_X(e)} = X_{a\cdot\Phi^t_X(e)}\,,\end{aligned}$$

where the second equality follows from the chain rule and the last one from X being left-invariant. □

Corollary 1.9. *Any left-invariant vector field X on G is complete, i.e., $\Phi^t_X(a)$ is defined for all $t \in R$ and all $a \in G$.*

Proof. Let $\varepsilon > 0$ be as in Lemma 1.8 and let $a \in G$. Suppose that

$$t_0 := \sup\{t \mid \Phi^{\cdot}_X(a) \text{ is defined at least until } t\} < \infty\,.$$

Let $b := \Phi^{t_0-\varepsilon/2}_X(a)$. By the previous lemma, $\Phi^t_X(b)$ is defined at least for $t \in (-\varepsilon, t_0 + \varepsilon/2)$, which is a contradiction to our assumption $t_0 < \infty$. □

Definition 1.10. *(i) A Lie group homomorphism $f : G \to H$ is a smooth group homomorphism between Lie groups G and H.*
(ii) A (real / quaternionic) representation is a Lie group homomorhism $f : G \to \mathrm{Gl}(V)$, where V is a complex (real / quaternionic) vector space.
(iii) A one-parameter subgroup in G is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e., α is smooth and satisfies $\alpha(s + t) = \alpha(s) \cdot \alpha(t)$ for all $s, t \in \mathbb{R}$.

Proposition 1.11. *The map $\{1\text{-parameter subgroups in } G\} \ni \alpha \mapsto \dot\alpha(0) \in T_eG$ is a bijection.*

Proof. Define

$$\Lambda : T_eG \cong \mathfrak{g} \ni X \mapsto (t \mapsto \Phi^t_X(e)) \in \{1\text{-parameter subgroups in } G\}$$
$$T_eG \ni \dot\alpha(0) \leftarrow\!\shortmid \alpha \in \{1\text{-parameter subgroups in } G\} : \Psi\,.$$

- $\Psi \circ \Lambda = \mathrm{id}$: $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Phi^t_X(e) = X_e$.
- $\Lambda \circ \Psi = \mathrm{id}$: We have to show that α is indeed the integral curve of the left-invariant vector field associated with $\dot\alpha(0)$:

$$\begin{aligned}\dot\alpha(t) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\alpha(t+s) = \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\alpha(t)\cdot\alpha(s) = \mathrm{d}(L_{\alpha(t)})_e\dot\alpha(0) \\ &= (\text{left-invariant vector field associated with } \dot\alpha(0))_{\alpha(t)}\,.\end{aligned}$$

□

Notation 1.12. *The Lie exponential map* $e^{\cdot} : \mathfrak{g} \to G$ *maps* $X \in T_eG \cong \mathfrak{g}$ *to* $e^{tX} := \Phi_X^t(e)$*. Thus,* $t \mapsto e^{tX}$ *is the 1-parameter subgroup in* G *associated with* X *as in Proposition 1.11.*

Proposition 1.13. *If* $X, Y \in \mathfrak{g}$*, then*

$$[X,Y]_e = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{tX} e^{sX} e^{-tX}.$$

Note that for fixed $t = t_0$, $s \mapsto e^{tX}e^{sX}e^{-tX}$ is a curve in G starting in $e \in G$, hence $t \mapsto \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{tX}e^{sX}e^{-tX}$ is a curve in T_eG.

Proof. Denote by $\mathcal{L}$ the Lie derivative. By its definition, we have

$$[X,Y]_e = (\mathcal{L}_X Y)_e = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{d}\left(\Phi_X^{-t}\right)_{\Phi_X^t(e)} Y_{\Phi_X^t(e)} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \Phi_X^{-t} \circ \Phi_Y^s \circ \Phi_X^t(e).$$

By Lemma 1.8 we have

$$\begin{aligned}\Phi_X^{-t}(\Phi_Y^s(\Phi_X^t(e))) &= \Phi_X^{-t}(\Phi_Y^s(e^{tX})) = \Phi_X^{-t}(\Phi_Y^s(e^{tX}\cdot e)) = \Phi_X^{-t}(e^{tX}\cdot \Phi_Y^s(e)) \\ &= e^{tX}\cdot \Phi_X^{-t}(e^{sY}) = e^{tX}\cdot e^{sY}\cdot e^{-tX}.\end{aligned}$$

□

Example 1.14. *Let* $G = \mathrm{Gl}(n;\mathbb{C}) \subseteq M(n,n;\mathbb{C})$*,* $e = E_n$ *the* $n\times n$ *identity matrix,* $C \in T_eG = M(n,n;\mathbb{C})$*,* $A \in G$*. Note that for small* t*,* $\det(E_n + tC) \neq 0$*, i.e.,* $E_n + tC \in \mathrm{Gl}(n;\mathbb{C})$*.*

$$\mathrm{d}(L_A)_e C = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} L_A(E_n + tC) = AC.$$

Hence, the left-invariant vector field X^C *associated with* C *is given by* $X^C_A = A\cdot C$ *.*

Next, we compute the Lie bracket of $C, D \in T_eG = M(n,n;\mathbb{C})$*. We have*

$$\begin{aligned}[C,D] &= [X^C, X^D]_e = \mathrm{d}(X^D)_e X^C_e - \mathrm{d}(X^C)_e X^D_e = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} X^D(E_n + tC) - \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} X^C(E_n + tD) \\ &= C\cdot D - D\cdot C\ ,\end{aligned}$$

where we have interpreted X^C *and* X^D *as maps from the open set* $\mathrm{Gl}(n;\mathbb{C}) \subseteq M(n,n;\mathbb{C})$ *to* $M(n,n;\mathbb{C}) \cong \mathbb{R}^{2n^2}$*, hence their Lie bracket is given by the difference of their directional derivatives with respect to each other.*

At last, we compute the Lie exponential map of G*. For* $C \in T_eG$*, the matrix exponential map* $t \mapsto \exp(tC) = E_n + tC + 1/2(tC)^2 + \ldots$ *is a 1-parameter subgroup in* G *(*$\exp((s+t)C) = \exp(sC)\cdot\exp(tC)$*) with* $\mathrm{d}/\mathrm{d}t_{|t=0}\exp(tC) = C$*, so it must be the one associated with* C*:*

$$e^{tC} = \exp(tC)\ .$$

The above formulae for $[C,D]$ *and* e^{tC} *also hold for any Lie subgroup of* G*!*

Lemma 1.15. *Let* $\Phi : G \to H$ *be a Lie group homomorphism.*

(i) $\Phi(e^{tX}) = e^{t\mathrm{d}\Phi_e X}$ *for all* $t \in \mathbb{R}$*,* $X \in T_eG$*.*
(ii) $[\mathrm{d}\Phi_e X, \mathrm{d}\Phi_e Y] = \mathrm{d}\Phi_e[X,Y]$*, hence* $\mathrm{d}\Phi_e T_eG \to T_eG$ *is a Lie algebra homomorphism , i.e., a vector space homomorphism which preserves Lie brackets.*

Proof. (i) We are done when we show that the left hand side is indeed a 1-parameter subgroup in H with the correct initial vector: $\Phi(e^{(s+t)X}) = \Phi(e^{sX}\cdot e^{tX}) = \Phi(e^{sX})\cdot\Phi(e^{tX})$ with initial vector $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Phi(e^{tX}) = \mathrm{d}\Phi_e(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} e^{tX}) = \mathrm{d}\Phi_e X$.

(ii)

$$\begin{aligned}[\mathrm{d}\Phi_e X, \mathrm{d}\Phi_e Y] &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{t\mathrm{d}\Phi_e X} e^{s\mathrm{d}\Phi_e X} e^{-t\mathrm{d}\Phi_e X} \overset{(i)}{=} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \Phi(e^{tX})\Phi(e^{sY})\Phi(e^{-tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{d}\Phi_e\left(\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{tX}e^{sY}e^{-tX}\right) = \mathrm{d}\Phi_e\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{tX}e^{sY}e^{-tX}\right) \\ &= \mathrm{d}\Phi_e([X,Y]),\end{aligned}$$

where we have used Proposition 1.13 in the first and last step.

□

Definition 1.16. *(i) For $a \in G$ let $I_a := L_a \circ R_a^{-1} : G \ni b \mapsto a \cdot b \cdot a^{-1} \in G$ be conjugation by a.*
(ii) For $a \in G$ let $\mathrm{Ad}_a := \mathrm{d}(I_a)_e : \mathfrak{g} \cong T_eG \to T_eG \cong \mathfrak{g}$.
(iii) For $X \in \mathfrak{g}$ let $\mathrm{ad}_X := [X, \cdot] : \mathfrak{g} \ni Y \mapsto [X, Y] \in \mathfrak{g}$.

Remark 1.17. *(i) I_a is a Lie group automorphism, i.e., a diffeomorphism and a group automorphism. Moreover, $\mathrm{Aut}(G)$ is a Lie group and $G \ni a \mapsto I_a \in \mathrm{Aut}(G)$ is a Lie group homomorphism.*
(ii) $\mathrm{Ad}_a : \mathfrak{g} \to \mathfrak{g}$ is by Lemma 1.15(ii) a Lie algebra automorphism. Moreoever, $\mathrm{Ad} : G \ni a \mapsto \mathrm{Ad}_a \in \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$ is a Lie group homomorphism, where $\mathrm{Gl}(\mathfrak{g})$ is the group of linear transformations of the vector space(!) $\mathfrak{g}$.
(iii) By the Jacobi-identity, we have $\mathrm{ad}_X[Y, Z] = [[X, Y], Z] + [Y, [X, Z]] = [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z]$. That is, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra derivation, i.e., a vector space endomorphism $\varphi \in \mathrm{End}(\mathfrak{g})$ with $\varphi[X, Y] = [\varphi X, Y] + [X, \varphi Y]$. Moreover, $\mathrm{ad} : \mathfrak{g} \ni X \mapsto \mathrm{ad}_X \in \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism, where the Lie bracket on $\mathrm{Der}(\mathfrak{g})$ is given by $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$ and $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

Lemma 1.18. *Let $X, Y \in \mathfrak{g} \cong T_eG$. Then*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{Ad}_{e^{tX}} = \mathrm{ad}_X\ .}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{Ad}_{e^{tX}} Y = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} I_{e^{tX}}(e^{sY}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} e^{tX} \cdot e^{sY} \cdot e^{-tX} \overset{1.13}{=} [X, Y] = \mathrm{ad}_X Y\,.$$

□

Corollary 1.19. *Apply Lemma 1.15(i) to $\Phi := \mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$:*

$$\mathrm{Ad}_{e^{tX}} = e^{t\,\mathrm{d}\,\mathrm{Ad}_e X} = e^{t\,\mathrm{ad}_X} = \exp(t\,\mathrm{ad}_X) = \mathrm{id} + t\,\mathrm{ad}_x + {}^{t^2}\!/_{2} t^2\,\mathrm{ad}_X^2 + \dots$$

and

$$\boxed{\mathrm{d}(\mathrm{Ad})_e = \mathrm{ad}\ .}$$

Summary .

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "e^{\cdot}"] & \mathrm{Der}(\mathfrak{g}) \arrow[r, "\subseteq"] \arrow[d, "e^{\cdot}=\exp"] & \mathrm{End}(\mathfrak{g}) \arrow[d, "e^{\cdot}=\exp"] \\
G \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(\mathfrak{g}) \arrow[r, "\subseteq"] & \mathrm{Gl}(\mathfrak{g})
\end{tikzcd}$$

1.2. Clifford Algebras.

Definition 1.20. *Let K be a field with $\operatorname{char} K \neq 2$, V a finite-dimensional K-vector space and q a quadratic form on V. We call (C, ι) a Clifford algebra for (V, q) if*
(i) C is an associative, unital K-algebra.
(ii) $\iota : V \to C$ is a K-linear map with

$$\iota(v)^2 = -q(v) \cdot \mathbb{1}_C \qquad \text{for all } v \in V\,.$$

(iii) If $\mathcal{A}$ is any associative, unital K-Algebra for which there is a map $j : V \to \mathcal{A}$ with

$$j(v)^2 = -q(v) \cdot \mathbb{1}_{\mathcal{A}} \qquad \text{for all } v \in V\,, \tag{1.1}$$

then there exists a unique K-algebra homomorphism $\tilde{j} : C \to \mathcal{A}$ such that

$$\begin{tikzcd}
 & C \arrow[dr, "\tilde{j}"] & \\
V \arrow[ur, "\iota"] \arrow[rr, "j"] & & \mathcal{A}
\end{tikzcd}$$

is commutative.

Proposition 1.21. *For any (V,q) there exists a Clifford algebra (C,ι) unique up to canonical isomorphism. Moreover, ι is injective and $\{\mathbb{1}_C\} \cup \iota(V) \subseteq C$ generates C.*

Proof. Let us first show uniqueness of the Clifford algebra. This is a standard argument using the universal property Definition 1.20(iii). Suppose we are given two Clifford algebras (C,ι) and (C',ι'). By definition, there exist unique maps $\widetilde{\iota} : C' \to C$ with $\widetilde{\iota} \circ \iota' = \iota$ and $\widetilde{\iota'} : C \to C'$ with $\widetilde{\iota'} \circ \iota = \iota'$. The map $\widetilde{\iota} \circ \widetilde{\iota'} : C \to C$ satisfies $\widetilde{\iota} \circ \widetilde{\iota'} \circ \iota = \widetilde{\iota} \circ \iota' = \iota$. Using Definition 1.20(iii) a third time, now with $\mathcal{A} = C$ and $j = \iota$, we see that $\mathrm{id}_C \circ \iota = \iota$. By uniqueness, we have $\widetilde{\iota} \circ \widetilde{\iota'} = \mathrm{id}_C$. Analogously, $\widetilde{\iota'} \circ \widetilde{\iota} = \mathrm{id}_{C'}$. Hence, (C,ι) is unique up to canonical isomorphism.

Next, we show that (C,ι) actually exists. Let $\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes^k}$ be the tensor algebra of V. Define $\mathcal{I}$ as the two-sided ideal generated by the set
$$\{v \otimes v + q(v) \mid v \in V\}$$
and $C := \mathcal{T}(V)/\mathcal{I}$. Let $\pi : \mathcal{T}(V) \to C$ be the canonical projection and define $\iota : V \hookrightarrow \mathcal{T}(V) \xrightarrow{\pi} C$, the concatenation of the injection $V \hookrightarrow \mathcal{T}(V)$ and the projection π.

Since $\mathcal{I}$ is a two-sided ideal, C inherits an associative, unital algebra structure from $\mathcal{T}(V)$. Furthermore, by the very definition of C and ι, we have $\iota(v)^2 = -q(v) \cdot \mathbb{1}_C$ for all $v \in V$.

Let now be $j : V \to \mathcal{A}$ be linear map into an associative, unital K-algebra with (1.1). By the universal property of the tensor algebra, j extends uniquely to a K-algebra homomorphism $\bar{j} : \mathcal{T}(V) \to \mathcal{A}$. Since j satisfies (1.1), we have $\mathcal{I} \subseteq \ker \bar{j}$. Hence, $\bar{j}$ descends uniquely to a map $\widetilde{j} : C \to \mathcal{A}$ satisfying $\widetilde{j} \circ \iota = j$.

To show that ι is injective, it suffices to prove that $V \cap \mathcal{I} = \{0\}$. This is a simple argument by induction over the degree of tensors. Finally, since $\mathcal{T}(V)$ is generated by V and $1 \in K = V^{\otimes^0}$, C is generated by $\iota(V)$ and $\mathbb{1}_C$. □

Remark 1.22. *(i) We will from now on denote the unique Clifford algebra associated with (V,q) by $(\mathcal{C}\ell(V,q),\iota)$ and view V as a subspace of $\mathcal{C}\ell(V,q)$ by virtue of ι. Moreoever, we will write $1 \in \mathcal{C}\ell(V,q)$ instead of $\mathbb{1}_{\mathcal{C}\ell(V,q)}$.*

(ii) If $b : V \times V \ni (v,w) \mapsto {}^1\!/\!_2(q(v+w) - q(v) - q(w)) \in K$ denotes the symmetric bilinear form associated with q, we have
$$\boxed{v \cdot w + w \cdot v = -2b(v,w) \cdot 1 \quad \textit{for all } v,w \in V}$$
in $\mathcal{C}\ell(V,q)$. In particular, if V has K-dimension n and $(e_1,\ldots,e_n)$ is a basis of V that diagonalizes b, then
$$\boxed{e_i^2 = -q(e_i) \quad \textit{for all} \quad i = 1,\ldots,n \quad \textit{and} \quad e_i \cdot e_j + e_j \cdot e_i = 0 \quad \textit{for all} \quad 1 \leqslant i \neq j \leqslant n\,.}$$
Definition 1.20(iii) says that $\mathcal{C}\ell(V,q)$ is the smallest associative, unital algebra containing V and satisfying these relations.

(iii) Let V,W be K-vector spaces, equipped with quadratic forms q and r, respectively. Applying Definition 1.20(iii) to $\iota_W \circ f$ for a K-linear map $f : V \to W$ which satisfies $f^*r = q$ (i.e. $r(f(v)) = q(v)$ for all $v \in V$) shows that f extends uniquely to an algebra homomorphism $\widetilde{f} : \mathcal{C}\ell(V,q) \to \mathcal{C}\ell(W,r)$. The uniqueness assertion in Definition 1.20(iii) also implies that, given another linear map $g : W \to U$ into a vector space U with a quadratic form s which satisfies $g^*s = r$, we have $\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}$.

Definition 1.23. *Let V be a K-vector space and $q : V \to K$ a quadratic form with associated symmetric bilinear form $b : V \times V \to K$.*

(i) Denote by $\alpha \in \mathrm{Aut}(\mathcal{C}\ell(V,q))$ the unique continuation of $-\mathrm{id}_V \in \mathrm{O}(b)$. Explicitely, $\alpha : \mathcal{C}\ell(V,q) \to \mathcal{C}\ell(V,q)$ is the unique K-linear map which satisfies
$$\boxed{\alpha(v_1 \cdot v_2 \cdots v_k) = \alpha(v_1) \cdot \alpha(v_2) \cdots \alpha(v_k) = (-1)^k v_1 \cdot v_2 \cdots v_k \quad \textit{for all} \quad k \in \mathbb{N}_0, v_1,\ldots,v_k \in V\,.}$$
In particular, $\alpha^2 = \mathrm{id}$.

(ii) For $i = 0,1$ define $\mathcal{C}\ell(V,q)^i := \{x \in \mathcal{C}\ell(V,q) \mid \alpha(x) = (-1)^i x\}$, i.e., $\mathcal{C}\ell(V,q)^i$ is the $(-1)^i$-eigenspace of α, and
$$\mathcal{C}\ell(V,q) = \mathcal{C}\ell(V,q)^0 \oplus \mathcal{C}\ell(V,q)^1\,.$$
Multiplication in $\mathcal{C}\ell(V,q)$ satisfies
$$\boxed{\mathcal{C}\ell(V,q)^i \cdot \mathcal{C}\ell(V,q)^j \subseteq \mathcal{C}\ell(V,q)^{i+j \bmod 2}\,.}$$

Remark 1.24. *(i) A K-algebra $\mathcal{A}$ with a splitting $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ such that multiplication in $\mathcal{A}$ obeys the rule $\mathcal{A}^i \cdot \mathcal{A}^j \subseteq \mathcal{A}^{i+j}$ is called a $\mathbb{Z}_2$-graded algebra . We call $\mathcal{A}^0$ the even part and $\mathcal{A}^1$ the odd part of $\mathcal{A}$ and we call* $\deg x := i$ *the degree of $x \in \mathcal{A}^i$. Note that $\mathcal{A}^0$ is always a subalgebra of $\mathcal{A}$.*

(ii) Given two $\mathbb{Z}_2$-graded K-algebras $\mathcal{A}$ and $\mathcal{B}$, their tensor product $\mathcal{A} \otimes \mathcal{B}$ is the K-algebra whose underlying K-vector space is the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ with multiplication $a \otimes b \cdot a' \otimes b' = a \cdot a' \otimes b \cdot b'$. Unfortunately, $\mathcal{A} \otimes \mathcal{B}$ is in general not *a $\mathbb{Z}_2$-graded algebra. To produce a $\mathbb{Z}_2$-graded algebra, we use the $\mathbb{Z}_2$-graded tensor product $\mathcal{A}\hat{\otimes}\mathcal{B}$ whose underlying vector space is again the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ and whose multiplication is defined on pure tensors of pure degree by*

$$a \otimes b \cdot a' \otimes b' = (-1)^{\deg b \cdot \deg a'} a \cdot a' \otimes b \cdot b' . \tag{1.2}$$

The $\mathbb{Z}_2$-grading of $\mathcal{A}\hat{\otimes}\mathcal{B}$ is given by

$$(\mathcal{A}\hat{\otimes}\mathcal{B})^0 = \mathcal{A}^0 \otimes \mathcal{B}^0 + \mathcal{A}^1 \otimes \mathcal{B}^1 ,$$
$$(\mathcal{A}\hat{\otimes}\mathcal{B})^1 = \mathcal{A}^0 \otimes \mathcal{B}^1 + \mathcal{A}^1 \otimes \mathcal{B}^0 .$$

Proposition 1.25. *Let V be a K-vector space with quadratic form q and associated symmetric bilinear form b. Assume we are given a b-orthogonal splitting $V = V_1 \oplus V_2$, i.e., $b(v_1, v_2) = 0$ for all $v_1 \in V_1, v_2 \in V_2$ (equivalently $q(v_1 + v_2) = q(v_1) + q(v_2)$). Then there is a natural isomorphism of Clifford algebras*

$$\mathcal{C}\ell(V, q) \to \mathcal{C}\ell(V_1, q_1)\hat{\otimes}\mathcal{C}\ell(V_2, q_2) ,$$

where $q_i := q_{|V_i} : V_i \to K$ is the restriction of q to V_i.

Proof. Define $j : V = V_1 \oplus V_2 \ni v_1 + v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2 \in \mathcal{C}\ell(V_1, q_1)\hat{\otimes}\mathcal{C}\ell(V_2, q_2)$. Then, we have for all $v_1 + v_2 \in V_1 \oplus V_2$ by (1.2)

$$\begin{aligned} j(v_1 + v_2)^2 &= (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2 + v_1 \otimes v_2 - v_1 \otimes v_2 = -q(v_1) \cdot 1 \otimes 1 - q(v_2) 1 \otimes 1 \\ &= -q(v_1 + v_2) 1 \otimes 1 . \end{aligned}$$

Hence, by Definition 1.20(iii), j extends uniquely to an algebra homomorphism $\tilde{j} : \mathcal{C}\ell(V, q) \to \mathcal{C}\ell(V_1, q_1)\hat{\otimes}\mathcal{C}\ell(V_2, q_2)$. To see that $\tilde{j}$ is bijective, we construct the inverse homomorphism. Let $g_i : V_i \to \mathcal{C}\ell(V, q)$, $i = 1, 2$, be the concatenation of the inclusion $V_i \hookrightarrow V$ and the inclusion $V \hookrightarrow \mathcal{C}\ell(V, q)$. Then g_i extends to an algebra homomorphism $\tilde{g}_i : \mathcal{C}\ell(V_i, q_i) \to \mathcal{C}\ell(V, q)$. The map $g : \mathcal{C}\ell(V_1, q_1)\hat{\otimes}\mathcal{C}\ell(V_2, q_2) \ni x \otimes y \mapsto \tilde{g}_1(x) \cdot \tilde{g}_2(y) \in \mathcal{C}\ell(V, q)$ is the inverse of $\tilde{j}$. It suffices to check this on pure tensors of vectors from V_1 and V_2, as those generate $\mathcal{C}\ell(V_1, q_1)\hat{\otimes}\mathcal{C}\ell(V_2, q_2)$ and hence determine g uniquely. □

Definition 1.26. *Let V be a K-vector space and q a quadratic form on V. Let $t : \mathcal{T}(V) \to \mathcal{T}(V)$ be the K-linear map given on pure tensors by*

$$t(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = v_k \otimes v_{k-1} \otimes \ldots \otimes v_1 .$$

*Then t preserves the ideal $\mathcal{I}$ from the proof of Proposition 1.21 and thus descends to a K-*linear *map*

$$\cdot^t : \mathcal{C}\ell(V, q) \to \mathcal{C}\ell(V, q) ,$$

the transpose . Note that $\cdot^t$ is an algebra antiautomorphism , i.e., $(x \cdot y)^t = y^t \cdot x^t$ for all $x, y \in \mathcal{C}\ell(V, q)$, and an involution, i.e., $(x^t)^t = x$ for all $x \in \mathcal{C}\ell(V, q)$.

With an eye on Riemannian manifolds we are interested in two particular Clifford algebras.

Notation 1.27. *Let $q_n : \mathbb{R}^n \ni x \mapsto \sum_{i=1}^n x_i^2 \in \mathbb{R}$ be the standard positive definite quadratic form on $\mathbb{R}^n$ and $q_n^{\mathbb{C}} : \mathbb{C}^n \ni z \mapsto \sum_{i=1}^n z_i^2 \in \mathbb{C}$ the standard quadratic form on $\mathbb{C}^n$. We let*

- $\mathcal{C}\ell_n = \mathcal{C}\ell(\mathbb{R}^n, q_n)$,
- $\mathbb{C}\ell_n = \mathcal{C}\ell(\mathbb{C}^n, q_n^{\mathbb{C}})$.

Remark 1.28. *It follows from Definition 1.20(iii) that the complexification $\mathcal{C}\ell_n \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathcal{C}\ell_n$, together with the complex extension of q_n, is isomorphic to $\mathbb{C}\ell_n$. In particular, from now on we will view $\mathcal{C}\ell_n$ as a subalgebra of $\mathbb{C}\ell_n$ and think of $\mathbb{C}\ell_n$ as $\mathcal{C}\ell_n$ with complex coefficients.*

Proposition 1.29. *There are algebra isomorphisms $\mathcal{C}\ell_n \cong \mathcal{C}\ell_{n+1}^0$ and $\mathbb{C}\ell_n \cong \mathbb{C}\ell_{n+1}^0$.*

Proof. Let $(e_1, \ldots, e_{n+1})$ be the standard basis of $\mathbb{R}^{n+1}$. Define $f : \mathbb{R}^n \to \mathcal{C}\ell^0_{n+1}$ by

$$f(e_i) := -e_i \cdot e_{n+1} \quad \text{for all} \quad 1 \leqslant i \leqslant n,$$

and linear extension. For $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n$ we have

$$\begin{aligned} f(x)^2 &= \left(-\sum_{i=1}^n x_i e_i \cdot e_{n+1} \right)^2 = \sum_{i,j=1}^n x_i x_j e_i \cdot e_{n+1} \cdot e_j \cdot e_{n+1} = -\sum_{i,j=1}^n x_i x_j e_i \cdot e_j \cdot e_{n+1} \cdot e_{n+1} \\ &= \sum_{i,j=1}^n x_i x_j e_i \cdot e_j = \left(\sum_{i=1}^n x_i e_i \right)^2 = x \cdot x = -q_n(x) \cdot 1 . \end{aligned}$$

By the universal property of Clifford algebras, f extends to an algebra homomorphism $\widetilde{f} : \mathcal{C}\ell_n \to \mathcal{C}\ell^0_{n+1}$. Evaluating $\widetilde{f}$ on a vector space basis of $\mathcal{C}\ell_n$ shows that it is an isomorphism (see Exercise no. 6). Finally, the isomorphism $\mathbb{C}\ell_n \cong \mathbb{C}\ell^0_{n+1}$ is the complexification of $\widetilde{f}$. □

Theorem 1.30. *For all $m \in \mathbb{N}$ there are algebra isomorphisms*

$$\Phi_{2m} : \mathbb{C}\ell_{2m} \to M(2,2;\mathbb{C}) \otimes M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C}) \cong M(2^m, 2^m; \mathbb{C}),$$

$$\Phi_{2m+1} : \mathbb{C}\ell_{2m+1} \to (M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C})) \oplus (M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C})) \cong M(2^m, 2^m; \mathbb{C}) \oplus M(2^m, 2^m; \mathbb{C}),$$

given as follows. Let $E := E_2$, $U := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$, $V := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $W := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. For $1 \leqslant j \leqslant m$ define

$$\phi_{2m} : \mathbb{C}^{2m} \ni e_{2j-1} \mapsto W \otimes W \otimes \ldots \otimes W \otimes \underset{j\text{-th slot}}{U} \otimes E \otimes \ldots \otimes E,$$

$$\phi_{2m} : \mathbb{C}^{2m} \ni e_{2j} \mapsto W \otimes W \otimes \ldots \otimes W \otimes \underset{j\text{-th slot}}{V} \otimes E \otimes \ldots \otimes E$$

and extend linearly. Then, $\phi_{2m}(x)^2 = -q^{\mathbb{C}}_{2m}(x) \cdot 1$ for all $x \in \mathbb{C}^{2m}$ and by the universal property of Clifford algebras, ϕ_{2m} extends to an algebra homomorphism Φ_{2m}, which turns out to be an isomorphism. To obtain Φ_{2m+1}, we define

$$\phi_{2m+1} : \mathbb{C}^{2m+1} \ni e_j \mapsto \begin{cases} (\phi_{2m}(e_j), \phi_{2m}(e_j)), & 1 \leqslant j \leqslant 2m, \\ (\mathrm{i}W \otimes \ldots \otimes W, -\mathrm{i}W \otimes \ldots \otimes W), & j = 2m+1, \end{cases}$$

and proceed analogously.

Definition 1.31. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\mathcal{A}$ be a finite-dimensional, associative and unital $\mathbb{K}$-algebra.*

(i) A representation of $\mathcal{A}$ is a $\mathbb{K}$-algebra homomorphism $\rho : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(V)$, where V is a finite-dimensional $\mathbb{K}$-vector space. In this situation, V is also called an $\mathcal{A}$-module. If the representation ρ is fixed, we shall write $x \cdot v := \rho(x)(v)$.
(ii) Given two representations $\rho : \mathcal{A} \to \mathrm{End}(V)$ and $\kappa : \mathcal{A} \to \mathrm{End}(W)$, their direct sum is the representation $\rho \oplus \kappa : \mathcal{A} \to \mathrm{End}(V \oplus W)$, given by $\rho \oplus \kappa(x)(v + w) = \rho(x)(v) + \kappa(x)(w)$.
(iii) A representation $\rho : \mathcal{A} \to \mathrm{End}(V)$ is called reducible if it is a direct sum $\rho = \rho_1 \oplus \rho_2 : \mathcal{A} \to End(V_1 \oplus V_2)$ with $V_i \neq \{0\}$, $i = 1, 2$. In other words, ρ is reducible if V splits into a nontrivial direct sum $V = V_1 \oplus V_2$ such that $\rho(x)(V_j) \subseteq V_j$ for all $x \in \mathcal{A}$, $j = 1, 2$. If ρ is not reducible, we call it irreducible .
(iv) Two representations $\rho : \mathcal{A} \to \mathrm{End}(V)$, $\kappa : \mathcal{A} \to \mathrm{End}(W)$ are called equivalent or isomorphic if there exists a $\mathbb{K}$-vector space isomorphism $F : V \to W$ such that $\rho(x) = F^{-1} \circ \kappa(x) \circ F$ for all $x \in \mathcal{A}$.
(v) We define modules, direct sums, irreducibility and equivalence analogously for representations of Lie groups.

Remark 1.32. *If $\rho : \mathcal{A} \to \mathrm{End}(V)$ is any representation of $\mathcal{A}$, then ρ can be decomposed into a direct sum $\rho = \rho_1 \oplus \ldots \oplus \rho_k$ of irreducible representations $\rho_i : \mathcal{A} \to \mathrm{End}(V_i)$. Indeed, we simply apply Definition 1.31(iii) recursively. This process must end by finite-dimensionality of V.*

For the next theorem, we need an important element in the Clifford algebras $\mathcal{C}\ell_n$ respectively $\mathbb{C}\ell_n$.

Definition 1.33. *Fix an orientation of $\mathbb{R}^n$ and let $(e_1, \ldots, e_n)$ be an oriented orthonormal basis w.r.t. $\langle \cdot, \cdot \rangle_{\mathrm{Eukl}}$. Define the volume element $\omega_n \in \mathcal{C}\ell_n$ by*

$$\omega_n := e_1 \cdot e_2 \cdots e_n,$$

and the complex volume element $\omega_n^{\mathbb{C}} \in \mathbb{C}\ell_n$ by

$$\omega_n^{\mathbb{C}} := \mathrm{i}^{\lfloor (n+1)/2 \rfloor} e_1 \cdot e_2 \cdots e_n = \mathrm{i}^{\lfloor (n+1)/2 \rfloor} \omega .$$

Here, $\lfloor x \rfloor$ denotes the largest integer which is smaller or equal to $x \in \mathbb{R}$.

Theorem 1.34. *There exists, up to equivalence, exactly one irreducible representation $\mathbb{C}\ell_{2m} \to \mathsf{End}_{\mathbb{C}}(V)$, where $\dim_{\mathbb{C}} V = 2^m$. There are, up to equivalence, exactly two irreducible representations $\rho : \mathbb{C}\ell_{2m+1} \to \mathsf{End}_{\mathbb{C}}(V)$, where $\dim V = 2^m$. These can be distinguished by the action of the complex volume element $\omega^{\mathbb{C}}_{2m+1}$, i.e., either $\rho(\omega^{\mathbb{C}}_{2m+1}) = +\,\mathrm{id}$ or $\rho(\omega^{\mathbb{C}}_{2m+1}) = -\,\mathrm{id}$.*

Proof. By Theorem 1.30, $\mathbb{C}\ell_{2m}$ is isomorphic to $M(2^m, 2^m; \mathbb{C})$. It is a classical fact that the only irreducible representation of $M(2^m, 2^m; \mathbb{C})$ is the standard one, given by matrix multiplication.

Again by Theorem 1.34, $\mathbb{C}\ell_{2m+1}$ is isomorphic to $M(2^m, 2^m; \mathbb{C}) \oplus M(2^m, 2^m; \mathbb{C})$. The two different representations are given by the standard representation of the first, respectively second, factor on $\mathbb{C}^{2^m}$.

For the proof of $\rho(\omega^{\mathbb{C}}_{2m+1}) = \pm\,\mathrm{id}$ and that these are inequalivalent representations, see Exercise no. 9. ☐

Proposition 1.35. *Let $\Phi_{2m} : \mathbb{C}\ell_{2m} \to M(2^m, 2^m; \mathbb{C}) \cong \mathsf{End}(\mathbb{C}^{2^m})$ be the irreducible representation given in Theorem 1.30 and $F : \mathbb{C}\ell_{2m-1} \to \mathbb{C}\ell^0_{2m}$ the algebra isomorphism from Proposition 1.29. Then the representation $\Phi_{2m} \circ F : \mathbb{C}\ell_{2m-1} \to M(2^m, 2^m; \mathbb{C}) \cong \mathsf{End}(\mathbb{C}^{2^m})$ is (equivalent to) the direct sum of the two irreducible representations of $\mathbb{C}\ell_{2m-1}$.*

Proof. See Exercise 10. ☐

1.3. **The Spin group, its Lie algebra and representations.**

Notation and Remarks 1.36. *Denote by $\mathcal{C}\ell_n^*$ the multiplicatively invertible elements of $\mathcal{C}\ell_n$. Then $\mathcal{C}\ell_n^*$ is an open subset of $\mathcal{C}\ell_n$ and hence a smooth manifold. Multiplication and inversion on $\mathcal{C}\ell_n^*$ are both smooth, hence $\mathcal{C}\ell_n^*$ is a Lie group.*

Definition 1.37. *(i) The Clifford group Γ_n of $\mathcal{C}\ell_n$ is the closed subgroup of $\mathcal{C}\ell_n^*$ given by*

$$\Gamma_n := \left\{ x \in \mathcal{C}\ell_n \mid \alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n \text{ for all } v \in \mathbb{R}^n \right\} .$$

(ii) Define the continuous group homomorphism $\lambda_n : \Gamma_n \to \mathrm{Gl}(n; \mathbb{R})$ by

$$\lambda_n(x)(v) := \alpha(x) \cdot v \cdot x^{-1} .$$

(iii) The norm of $\mathcal{C}\ell_n$ is the map

$$N : \mathcal{C}\ell_n \ni x \mapsto x \cdot \alpha(x^t) = x \cdot \alpha(x)^t \in \mathcal{C}\ell_n .$$

Remark 1.38. *(i) The maps $\alpha, \cdot^t : \mathcal{C}\ell_n \to \mathcal{C}\ell_n$ leave Γ_n invariant. Indeed, if $x \in \Gamma_n$, then $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$ and by definition of α we have*

$$\alpha(\alpha(x)) \cdot v \cdot \alpha(x)^{-1} = -\alpha(\alpha(x)) \cdot \alpha(v) \cdot \alpha(x)^{-1} = -\alpha(\alpha(x) \cdot v \cdot x^{-1}) = \alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$$

for all $v \in \mathbb{R}^n$ and analogously for $\cdot^t$.

(ii) Note that for $x \in \mathbb{R}^n$ we have $N(x) = x \cdot \alpha(x^t) = x \cdot \alpha(x) = -x \cdot x = q_n(x) = \|x\|^2$.

Lemma 1.39. *The kernel of the group homomorphism $\lambda_n : \Gamma_n \to \mathrm{Gl}(n; \mathbb{R})$ is $\ker \lambda_n = \mathbb{R}^* \cdot 1$.*

Proof. Let $x \in \ker \lambda_n$. Then by definition $\alpha(x) \cdot v \cdot x^{-1} = v$ for all $v \in \mathbb{R}^n$, which is equivalent to

$$\alpha(x) \cdot v = v \cdot x \quad \text{for all} \quad v \in \mathbb{R}^n .$$

We decompose x into its even and odd part, $x = x^0 + x^1$ with $x^i \in \mathcal{C}\ell_n^i$. Then the above statement is equivalent to

$$x^0 \cdot v = v \cdot x^0 \quad \text{and} \quad -x^1 \cdot v = v \cdot x^1 \quad \text{for all} \quad v \in \mathbb{R}^n . \tag{1.3}$$

Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. We express x^0 as a linear combination of the basis vectors from Exercise 6 and write

$$x^0 = a + e_1 b ,$$

where $a \in \mathcal{C}\ell_n^0, b \in \mathcal{C}\ell_n^1$ and neither a nor b contain a term with a factor e_1. We apply the first relation in (1.3) to $v = e_1$ and obtain

$$(a + e_1 b) e_1 = e_1 (a + e_1 b) .$$

Since a has even degree and contains no term with a factor e_1 we have $a e_1 = e_1 a$. Analogously, we have $e_1 b = -b e_1$. Hence,

$$a + e_1 b = a - e_1 b ,$$

which in turn implies $e_1 b = 0$ and x^0 contains to term with a factor e_1. By applying the same argument to e_i, $i = 2, \ldots, n$, we conclude that x^0 is a linear combination of the elements from Exercise 6 no term of which contains a factor e_i, i.e., $x^0 \in \mathbb{R} \cdot 1$.

Proceeding analogously with the second relation in (1.3) shows $x^1 \in \mathbb{R} \cdot 1$. But since $1 \in \mathcal{C}\ell_n^0$ we must have $x^1 = 0$. Hence $x = x^0 \in \mathbb{R} \cdot 1 \cap \Gamma_n = \mathbb{R}^* \cdot 1$. □

Lemma 1.40. *If $x \in \Gamma_n$, then $N(x) \in \mathbb{R}^*$ and the restriction $N_{|\Gamma_n} : \Gamma_n \to \mathbb{R}^*$ is a group homomorphism with $N(\alpha(x)) = N(x)$ for all $x \in \Gamma_n$.*

Proof. Let $x \in \Gamma_n$. Then $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$. Since the transpose acts as the identity on $\mathbb{R}^n$, we get $(x^t)^{-1} \cdot v \cdot \alpha(x)^t = \alpha(x) \cdot v \cdot x^{-1}$. Thus, $v = x^t \cdot \alpha(x) \cdot v \cdot (\alpha(x)^t \cdot x)^{-1} = \alpha(x)^t \cdot x \cdot v \cdot (\alpha(x)^t \cdot x)^{-1}$ which implies that $\alpha(x)^t \cdot x \in \ker \lambda_n$. By Remark 1.38(i), $y = \alpha(x)^t \in \Gamma_n$ and by what we just showed $\alpha(y)^t \cdot y = \alpha(\alpha(x)^t)^t \cdot \alpha(x)^t = x \cdot \alpha(x)^t = N(x) \in \ker \lambda_n$. By the last lemma, $N(x) \in \mathbb{R}^* \cdot 1$.

To show that N restricted to Γ_n is a homomorphism, note that $\mathbb{R} \cdot 1$ is central in $\mathcal{C}\ell_n$. Hence, for $x, y \in \Gamma_n$, we have $N(x \cdot y) = x \cdot y \cdot \alpha(x \cdot y)^t = x \cdot y \cdot \alpha(y)^t \cdot \alpha(x)^t = xN(y)\alpha(x)^t = x \cdot \alpha(x)^t N(y) = N(x)N(y)$.

At last, we have $N(\alpha(x)) = \alpha(x) \cdot \alpha(\alpha(x))^t = \alpha(x \cdot \alpha(x)^t) = \alpha(N(x)) = N(x)$. □

Proposition 1.41. *We have*

(i) $\mathbb{R}^n \backslash \{0\} \subseteq \Gamma_n$,

(ii) for $x \in \mathbb{R}^n \backslash \{0\}$, $\lambda_n(x) \in \mathrm{Gl}(n; \mathbb{R})$ is the reflection about the hyperplane $x^\perp$. In particular, $\lambda_n(\Gamma_n) \subseteq \mathrm{O}(n)$, the orthogonal group.

Proof. Let $x \in \mathbb{R}^n \backslash \{0\}$. By Lemma 1.39, $\lambda_n(x) = \lambda_n(\|x\| \cdot \frac{x}{\|x\|}) = \lambda_n(\frac{x}{\|x\|})$, which is why we can assume w.l.o.g. that $\|x\| = 1$. Choose an orthonormal basis $(e_1 = x, e_2, \ldots, e_n)$ of $\mathbb{R}^n$. Then, for $v = \sum_{i=1}^n a_i e_i$ we have by the Clifford relations

$$\begin{aligned} \lambda_n(x)(v) = \lambda_n(e_1)\left(\sum_{i=1}^n a_i e_i\right) &= \sum_{i=1}^n a_i \alpha(e_1) \cdot e_i \cdot e_1^{-1} = -\sum_{i=1}^n a_i e_1 \cdot e_i \cdot e_1^{-1} \\ &= -a_1 e_1 - \sum_{i=2}^n a_i e_1 \cdot e_i \cdot e_1^{-1} = -a_1 e_1 + \sum_{i=2}^n a_i e_i \in \mathbb{R}^n \,. \end{aligned}$$

In particular, $\lambda_n(x)$ is the reflection about $x^\perp$ and $\|\lambda_n(x)(v)\| = \|v\|$. □

Definition 1.42. *The Pin group $\mathrm{Pin}(n) \subseteq \mathcal{C}\ell_n^*$ is the kernel of $N : \Gamma_n \to \mathbb{R}^*$. The Spin group $\mathrm{Spin}(n)$ is the group $\mathrm{Pin}(n) \cap \mathcal{C}\ell_n^0$.*

Theorem 1.43. *(i) The Pin and Spin groups are Lie groups explicitely given by*

$$\begin{aligned} \mathrm{Pin}(n) &= \{v_1 \cdot v_2 \cdots v_k \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, 0 \leqslant i \leqslant k, k \in \mathbb{N}_0\} \,, \\ \mathrm{Spin}(n) &= \{v_1 \cdot v_2 \cdots v_{2k} \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, 0 \leqslant i \leqslant k, k \in \mathbb{N}_0\} \,. \end{aligned}$$

(ii) $\lambda_{n|\mathrm{Pin}(n)} : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is a surjective Lie group homomorphism with kernel $\{\pm 1\}$.

(iii) $(\lambda_{n|\mathrm{Pin}(n)})^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n)$.

(iv) $\mathrm{Spin}(n)$ is connected for $n \geqslant 2$.

Proof. Recall that any orthogonal map $A \in \mathrm{O}(n)$ can be written as the concatenation $A_{v_1} \circ \ldots \circ A_{v_k}$ of reflections A_{v_i} about hyperplanes $v_i^\perp$, where $v_i \in \mathbb{R}^n$ with $\|v_i\| = 1$. By Proposition 1.41 and the definition of $\mathrm{Pin}(n)$, $v_1 \cdots v_k \in \mathrm{Pin}(n)$ and $\lambda_n(v_1 \cdots v_k) = \lambda_n(v_1) \cdots \lambda_n(v_k) = A_{v_1} \circ \ldots \circ A_{v_k} = A$. Furthermore, the kernel of $\lambda_{n|\mathrm{Pin}(n)}$ is $\ker \lambda_n \cap \ker N = \{x \in \mathbb{R}^* \cdot 1 \mid N(x) = 1\} = \{\pm 1\}$, which also shows the explicit expression for $\mathrm{Pin}(n)$.

Recall also that the group $\mathrm{SO}(n) \subseteq \mathrm{O}(n)$ can be characterized as the group of maps which can be written as the concatenation of an even number of reflections. This shows (iii) and the explicit expression for $\mathrm{Spin}(n)$.

To see that $\mathrm{Pin}(n)$ is a Lie group, recall that Γ_n is a closed subgroup of the Lie group $\mathcal{C}\ell_n^*$. It is a theorem (see, e.g., Lee, J. M. *Introduction to smooth manifolds*) that any algebraic subgroup of a Lie group which is topologically closed is automatically a Lie group in its own right. This makes Γ_n into a Lie group and $N : \Gamma_n \to \mathbb{R}^*$ a Lie group homomorphism. Now $\mathrm{Pin}(n)$ is the kernel of N, which makes it a topologically closed algebraic subgroup and therefore a Lie group. Similarly, $\mathrm{Spin}(n)$ is the inverse image of the Lie group $\mathrm{SO}(n)$ and therefore, again, a topologically closed algebraic subgroup, hence a Lie group. The map $\lambda_{n|\mathrm{Pin}(n)}$ is the concatenation of multiplication, inversion and the (restriction of the) linear map α, hence smooth and therefore a Lie group homomorphism.

In light of (iii), it suffices to connect -1 to 1 with an arc in $\mathrm{Spin}(n)$ to see (iv). Such an arc is

$$c : [0,\pi] \ni t \mapsto \cos(t) + \sin(t) e_1 \cdot e_2 = (\sin\tfrac{t}{2} e_1 - \cos\tfrac{t}{2} e_2)(\sin\tfrac{t}{2} e_1 + \cos\tfrac{t}{2} e_2) \in \mathrm{Spin}(n)\,.$$

□

Remark 1.44. *We will henceforth only be interested in the Spin group and will from now on view λ_n as a map*

$$\lambda := \lambda_n : \mathrm{Spin}(n) \to SO(n)$$
$$g \mapsto (v \mapsto \alpha(g) \cdot v \cdot g^{-1} = g \cdot v \cdot g^{-1})\,.$$

For the next proposition, recall that the Lie group $\mathcal{C}\ell_n^*$ is an open subset of $\mathcal{C}\ell_n$. Hence, $T_1 \mathcal{C}\ell_n^* = \mathcal{C}\ell_n$. Since $\mathrm{Spin}(n)$ is a submanifold of $\mathcal{C}\ell_n^*$, $T_1 \mathrm{Spin}(n)$ is a subspace of $\mathcal{C}\ell_n$.

Proposition 1.45. *The tanget space to* $\mathrm{Spin}(n)$ *at 1 is*

$$T_1 \mathrm{Spin}(n) = \mathrm{span}_{\mathbb{R}}\{e_i \cdot e_j \,|\, 1 \leqslant i < j \leqslant n\} \subseteq \mathcal{C}\ell_n\,.$$

Proof. For $1 \leqslant i < j \leqslant n$, consider the curve

$$\gamma : \mathbb{R} \ni t \mapsto \cos(t) + \sin(t) e_i \cdot e_j = (\sin\tfrac{t}{2} e_i - \cos\tfrac{t}{2} e_j)(\sin\tfrac{t}{2} e_i + \cos\tfrac{t}{2} e_j) \in \mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n\,.$$

We have $\gamma(0) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\gamma(t) = e_i \cdot e_j$. This shows "$\supseteq$". By Exercise 6, the stated subset of $\mathcal{C}\ell_n$ clearly has dimension $\frac{1}{2}n(n-1)$. But from Theorem 1.43, we already know that $\dim T_1 \mathrm{Spin}(n) = \dim \mathrm{Spin}(n) = \dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$, showing "$\subseteq$". □

Corollary 1.46. *The Lie algebra of* $\mathrm{Spin}(n)$ *is*

$$\mathfrak{spin}(n) \cong \mathrm{span}_{\mathbb{R}}\{e_i \cdot e_j \,|\, 1 \leqslant i < j \leqslant n\} \subseteq \mathcal{C}\ell_n$$

with the Lie bracket $[x,y] = x \cdot y - y \cdot x$.

Proof. Following Example 1.14, one checks that the Lie algebra of $\mathcal{C}\ell_n^*$ is $\mathcal{C}\ell_n$ equipped with the Lie bracket $[x,y] = x \cdot y - y \cdot x$. The Lie algebra of $\mathrm{Spin}(n)$ then inherits this Lie bracket. □

Proposition 1.47. *The differential* $\lambda_* = \mathrm{d}\lambda_e : T_1 \mathrm{Spin}(n) \cong \mathfrak{spin}(n) \to \mathfrak{so}(n) \cong T_{E_n} \mathrm{SO}(n)$ *is an isomorphism explicitely given by*

$$\lambda_*(e_i \cdot e_j) = 2X_{e_i,e_j}\,,$$

where X_{e_i,e_j} are the matrices from Exercise 5.

Proof. Since $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a surjective Lie group homomorphism between Lie groups of equal dimension, its differential at 1 must be an isomorphism. We consider again for $1 \leqslant i < j \leqslant n$ the path $\gamma : \mathbb{R} \ni t \mapsto \cos(t) + \sin(t) e_i \cdot e_j \in \mathrm{Spin}(n)$. Note that $\gamma(t)^{-1} = \gamma(-t)$. Hence, for $v = \sum_{k=1}^n v_k e_k \in \mathbb{R}^n$ we have

$$\begin{aligned}
\lambda_*(e_i \cdot e_j)(v) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \lambda(\gamma(t))(v) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \gamma(t) \cdot v \cdot \gamma(t)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \gamma(t) \cdot v \cdot \gamma(-t) = e_i \cdot e_j \cdot v - v \cdot e_i \cdot e_j \\
&= v_i(e_i \cdot e_j \cdot e_i - e_i \cdot e_i \cdot e_j) + v_j(e_i \cdot e_j \cdot e_j - e_j \cdot e_i \cdot e_j) + \sum_{k \neq i,j} v_k(e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j) \\
&= 2v_i e_j - 2v_j e_i = 2(v_i e_j - v_j e_i) = 2X_{e_i,e_j} v\,.
\end{aligned}$$

□

Definition 1.48. *The (complex) fundamental spin representation of* $\mathrm{Spin}(n)$ *is the Lie group homomorphism*

$$\kappa_n : \mathrm{Spin}(n) \to \mathrm{Gl}(\Sigma_n)$$

given by restricting an irreducible complex representation $\mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$ to $\mathrm{Spin}(n) \subseteq \mathbb{C}\ell_n^0 \subseteq \mathbb{C}\ell_n$. We call Σ_n the spinor module and an element $s \in \Sigma_n$ a spinor .

Proposition 1.49. *When n is odd the definition of the complex spin representation is independent of which irreducible representation of $\mathbb{C}\ell_n$ was used. In particular, it is well-defined. Moreover, when n is odd, κ_n is irreducible.*

When n is even, there is a decomposition

$$\kappa_n = \kappa_n^+ \oplus \kappa_n^- , \qquad \kappa_n^\pm : \mathrm{Spin}(n) \to \mathrm{Gl}(\Sigma_n^\pm)$$

into irreducible representations $\kappa_n^\pm$ called the positive respectively negative half-spin representations. Accordingly, the modules $\Sigma_n^\pm$ are the positive respectively negative half-spinor modules.

Proof. Let $n = 2m+1$. Recall from Theorem 1.34 that $\mathbb{C}\ell_{2m+1}$ has two irreducible representations $\rho_i : \mathbb{C}\ell_{2m+1} \to \mathrm{Gl}(V)$, $i = 1, 2$, which can be distinguished by $\rho_1(\omega^{\mathbb{C}}_{2m+1}) = +\mathrm{id}$ and $\rho_2(\omega^{\mathbb{C}}_{2m+1}) = -\mathrm{id}$. Since α is an algebra automorphism of $\mathbb{C}\ell_{2m+1}$, $\rho_2 \circ \alpha$ is also a representation of $\mathbb{C}\ell_{2m+1}$ with $\rho_2 \circ \alpha(\omega^{\mathbb{C}}_{2m+1}) = \rho_2(-\omega^{\mathbb{C}}_{2m+1}) = -\rho_2(\omega^{\mathbb{C}}_{2m+1}) = +\mathrm{id}$, so ρ_1 and $\rho_2 \circ \alpha$ are equivalent. Now recall that $\mathbb{C}\ell^0_{2m+1}$ is the $(+1)$-eigenspace of α, hence ρ_1 and ρ_2 are equivalent when restricted to $\mathbb{C}\ell^0_{2m+1}$.

By Proposition 1.29 there is an algebra isomorphism $F : \mathbb{C}\ell_{2m} \to \mathbb{C}\ell^0_{2m+1}$. Since $\rho_i \circ F : \mathbb{C}\ell_{2m} \to \mathrm{Gl}(V)$ is a nontrivial complex representation of $\mathbb{C}\ell_{2m}$ of dimension 2^m, it must be the unique irreducible one, hence $\rho = \rho_{i|\mathbb{C}\ell^0_{2m+1}}$ is an irreducible representation of $\mathbb{C}\ell^0_{2m+1}$.

To see that $\rho_{|\mathrm{Spin}(2m+1)}$ is an irreducible Lie group representation, assume that $\rho_{|\mathrm{Spin}(2m+1)}$ splits into the direct sum of two nontrivial representations, i.e., there exists a nontrivial splitting $V = W_1 \oplus W_2$ such that $\rho(x)(W_j) \subseteq W_j$ for all $x \in \mathrm{Spin}(2m+1)$. By Exercise 6 and Theorem 1.43(i), $\mathrm{Spin}(2m+1)$ contains an additive basis $e_{i_1} \cdot e_{i_2} \cdots e_{i_{2k}}$, $1 \leqslant i_1 < i_2 < \ldots < i_{2k} \leqslant 2m+1$ of $\mathbb{C}\ell^0_{2m+1}$. Since ρ is the restriction to $\mathrm{Spin}(2m+1)$ of an irreducible representation of $\mathbb{C}\ell^0_n$, not all of these basis elements leave W_j invariant, i.e., there exists $1 \leqslant i_1 < i_2 < \ldots < i_{2k} \leqslant 2m+1$ and $j \in \{1, 2\}$ such that $\rho(e_{i_1} \cdot e_{i_2} \cdots e_{i_{2k}})(W_j) \nsubseteq W_j$. A contradiction. Hence, $\rho_{\mathrm{Spin}(2m+1)}$ is an irreducible representation of $\mathrm{Spin}(2m+1)$.

Now let $n = 2m$. There is exactly one irreducible representation $\rho : \mathbb{C}\ell_{2m} \to \mathrm{Gl}(V)$ of $\mathbb{C}\ell_{2m}$. If we restrict ρ to $\mathbb{C}\ell^0_{2m}$, then Proposition 1.35 tells us that $\rho_{|\mathbb{C}\ell^0_{2m}}$ splits into the direct sum of two inequivalent irreducible representations. We argue as in the case $n = 2m+1$ that their restrictions to $\mathrm{Spin}(2m) \subseteq \mathbb{C}\ell^0_{2m}$ are irreducible Lie group representations. □

Remark 1.50. *The fundamental spin representation is not induced by a representation of $\mathrm{SO}(n)$ (through λ). Indeed, $-1 \in \mathrm{Spin}(n)$ and $\kappa_n(-1) = -\mathrm{id}_{\Sigma_n}$ while for every representation $\rho : \mathrm{SO}(n) \to \mathrm{Gl}(V)$ we have $\rho \circ \lambda(-1) = \rho(E_n) = \mathrm{id}_V$.*

Proposition 1.51. *Let $\rho : \mathbb{C}\ell_n \to \mathrm{Gl}(V)$ be an irreducible representation of the complex Clifford algebra $\mathbb{C}\ell_n$. Then there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that*

$$\langle \rho(x)(v), \rho(x)(w) \rangle = \langle v, w \rangle \qquad \text{for all} \quad x \in \mathbb{R}^n \subseteq \mathbb{C}\ell_n \quad \text{with} \quad \|x\| = 1 \quad \text{and all} \quad v, w \in V\,. \tag{1.4}$$

In particular,

(i) multiplication by unit vectors is skew-symmetric, i.e., for all $x \in \mathbb{R}^n$ with $\|x\| = 1$ and all $v, w \in V$ we have

$$\langle \rho(x)(v), w \rangle = \langle \rho(x)^2(v), \rho(x)(w) \rangle = \langle \rho(x^2)(v), \rho(x)(w) \rangle = -\langle v, \rho(x)(w) \rangle,$$

(ii) there exists a $\mathrm{Spin}(n)$-invariant inner product $\langle \cdot, \cdot \rangle$ on Σ_n, i.e., $\langle \kappa_n(g)(\sigma), \kappa_n(g)(\tau) \rangle = \langle \sigma, \tau \rangle$ for all $g \in \mathrm{Spin}(n)$ and $\sigma, \tau \in \Sigma_n$. In short: $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$.

Proof. Since ρ is an irreducible representation, there exists a linear isomorphism $F : V \to \mathbb{C}^{2^{n/2}}$ such that $\rho(\cdot) = F^{-1} \circ \Phi_n(\cdot) \circ F$ in case $n = 2m$ or $\rho(\cdot) = F^{-1} \circ \pi_i \circ \Phi_n(\cdot) \circ F$ if $n = 2m+1$, where Φ_n is the algebra isomorphism from Theorem 1.30 and π_i, $i = 1, 2$, the projection on the first respectively second factor.

We define the inner product $\langle \cdot, \cdot \rangle$ on V to be the pullback $\langle v, w \rangle := (F(v), F(w))$ of the standard hermitian inner product

$$(a, b) = \sum_{i=1}^{2^{n/2}} a_i \overline{b}_i$$

on $\mathbb{C}^{2^{n/2}}$. Then (1.4) follows from the matrices U, V and W from Theorem 1.30 being unitary. □

Definition 1.52. *(i) A Clifford multiplication is a complex linear map*

$$\begin{aligned} \mu : \mathbb{R}^n \otimes \Sigma_n &\to \Sigma_n \\ x \otimes \sigma &\mapsto x \cdot \sigma := \mu(x \otimes \sigma) \end{aligned}$$

which satisfies

$$x \cdot (y \cdot \sigma) + y \cdot (x \cdot \sigma) = -2\langle x, y\rangle \cdot \sigma \qquad \text{for all} \quad x, y \in \mathbb{R}^n, \sigma \in \Sigma_n .$$

(ii) Two Clifford multiplications $\mu_1, \mu_2 : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ are equivalent if there exists a vector space isomorphism $F : \Sigma_n \to \Sigma_n$ such that

$$\mu_1(x \otimes \sigma) = F^{-1}(\mu_2(x \otimes F(\sigma))) \qquad \text{for all} \quad x \in \mathbb{R}^n, \sigma \in \Sigma_n .$$

Proposition 1.53. *If n is even then there exists up to equivalence exactly one Clifford multiplication. If n is odd there exist up to equivalence exactly two Clifford multiplications one of which is the negative of the other. They can be distinguished by the action of the complex volume element $\omega_n^{\mathbb{C}}$, i.e., they satisfy*

$$\omega_n^{\mathbb{C}} \cdot \sigma := i^{\lfloor (n+1)/2 \rfloor} e_1 \cdot (e_2 \cdot (\dots (e_n \cdot \sigma))) = \pm\sigma \qquad \text{for all} \quad \sigma \in \Sigma_n .$$

Proof. If $\rho : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$ is an irreducible representation then $\mu(x \otimes \sigma) := \rho(x)(\sigma)$ is a Clifford multiplication. This shows existence and in case n is odd that there are two Clifford multiplications which can be distinguished by the action of the complex volume element.

To see uniqueness, let $\mu : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ be a Clifford multiplication. Define $\rho : \mathbb{R}^n \to \mathsf{End}(\Sigma_n)$ by $\rho(x)(\sigma) := \mu(x \otimes \sigma)$. Then $\rho(x)^2 = -\|x\|^2 \cdot \mathrm{id}_{\Sigma_n}$. Hence, ρ extends uniquely to an algebra homomorphism $\tilde{\rho} : \mathcal{C}\ell_n \to \mathsf{End}(\Sigma_n)$ and by complexification to an algebra homomorphism $\tilde{\rho} : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$. Since $\dim \Sigma_n = 2^{n/2}$, $\tilde{\rho}$ must be an irreducible representation. This completes the proof. □

Corollary 1.54. *Every Clifford multiplication satisfies*

(i) $\langle x \cdot \sigma, x \cdot \tau\rangle = \langle \sigma, \tau\rangle$ and

(ii) $\langle x \cdot \sigma, \tau\rangle = -\langle \sigma, x \cdot \tau\rangle$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$ and all $\sigma, \tau \in \Sigma_n$, where $\langle\cdot,\cdot\rangle$ is the $\mathrm{Spin}(n)$-invariant inner product on Σ_n.

Remark 1.55. *The group $\mathrm{Spin}(n)$ acts on Σ_n by the fundamental spin representation $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$ and on $\mathbb{R}^n$ by $\lambda : \mathrm{Spin}(n) \to \mathrm{O}(n)$. If we form the tensor product $\mathbb{R}^n \otimes \Sigma_n$, then $\mathrm{Spin}(n)$ acts thereon via the tensor representation*

$$\begin{aligned} \lambda \otimes \kappa_n : \mathrm{Spin}(n) &\to \mathrm{U}(\mathbb{R}^n \otimes \Sigma_n) \\ g &\mapsto (x \otimes \sigma \mapsto \lambda(g)(x) \otimes \kappa_n(g)(\sigma)) . \end{aligned}$$

Proposition 1.56. *Every Clifford multiplication $\mu : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ is $\mathrm{Spin}(n)$-equivariant, i.e., we have*

$$\mu(\lambda \otimes \kappa_n(g)(x \otimes \sigma)) = \kappa_n(g)(\mu(x \otimes \sigma)) \qquad \text{for all} \quad g \in \mathrm{Spin}(n), x \in \mathbb{R}^n, \sigma \in \Sigma_n .$$

Put differently, the diagram

$$\begin{tikzcd} \mathbb{R}^n \otimes \Sigma_n \arrow[r, "\mu"] \arrow[d, "\lambda\otimes\kappa_n"] & \Sigma_n \arrow[d, "\kappa_n"] \\ \mathbb{R}^n \otimes \Sigma_n \arrow[r, "\mu"] & \Sigma_n \end{tikzcd}$$

is commutative.

Remark. *The following proof actually shows: If we choose one representative μ from the given equivalence class of Clifford multiplications, then there exists precisely one representative κ_n of the equivalence class of the fundamental spin representation such that μ is $\mathrm{Spin}(n)$-equivariant w.r.t. $\lambda \otimes \kappa_n$ and κ_n.*

Proof. The Clifford multiplication μ satisfies $\mu(x \otimes \sigma) = \rho(x)(\sigma)$ where $\rho : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$ is an irreducible representation. We also have $\kappa = \rho_{|\,\mathrm{Spin}(n)}$. The claim is now a straightforward calculation:

$$\begin{aligned} \mu(\lambda \otimes \kappa_n(g)(x \otimes \sigma)) &= \mu(\lambda(g)(x) \otimes \kappa_n(g)(\sigma)) = \mu(g \cdot x \cdot g^{-1} \otimes \rho(g)(\sigma)) \\ &= \rho(g \cdot x \cdot g^{-1})(\rho(g)(\sigma)) = \rho(g \cdot x \cdot g^{-1} \cdot g)(\sigma) = \rho(g \cdot x)(\sigma) = \rho(g) \circ \rho(x)(\sigma) \\ &= \kappa_n(g)(\mu(x \otimes \sigma)) . \end{aligned}$$

□

Remark 1.57. *Since there is no ambiguity about how* $\mathrm{Spin}(n)$ *acts on* Σ_n *(via* κ_n*) respectively* $\mathbb{R}^n$ *(via* λ*), we can abbreviate notation and simply write* $g\sigma$ *respectively* gx *for all* $g \in \mathrm{Spin}(n)$, $x \in \mathbb{R}^n$ *and* $\sigma \in \Sigma_n$.

The equivariance of Clifford multiplication can now be stated very concisely:

$$gx \cdot g\sigma = g(x \cdot \sigma) \qquad \text{for all} \quad g \in \mathrm{Spin}(n), x \in \mathbb{R}^n, \sigma \in \Sigma_n\,.$$

In fact, using this shorthand notation, the proof of Proposition 1.56 becomes very short:

$$gx \cdot g\sigma = g \cdot x \cdot g^{-1} \cdot g\sigma = g \cdot x \cdot \sigma = g(x \cdot \sigma)\,.$$

Note, however, that it is not easy to unravel what exactly is happening here.

2. Intermezzo: Gauge theory

Definition 2.1. *Let* P *be a smooth manifold and* G *a Lie group.*

(i) A (right-)action of G *on* P *is a smooth map*

$$P \times G \ni (p, g) \mapsto p \cdot g \in P$$

such that
- $p \cdot e = p$ *for all* $p \in P$ *and*
- $(p \cdot g) \cdot h = p \cdot (g \cdot h)$ *for all* $g, h \in G$ *and* $p \in P$.

For $g \in G$ *the map* $R_g : P \in p \mapsto p \cdot g \in P$ *is called right-translation by* g. R_g *is a diffeomorphism with inverse* $R_g^{-1} = R_{g^{-1}}$.

(ii) A right action of G *on* P *is*
- *free if* $p \cdot g = g$ *for* $p \in P$ *and* $g \in G$ *implies* $g = e$*, i.e., the only right translation that has fixed points is* R_e,
- *transitive if for all* $p, q \in P$ *there exists a* $g \in G$ *such that* $p \cdot g = q$,
- *simply-transitive if it is free and transitive, i.e., if for all* $p, q \in P$ *there exists precisely one* $g \in G$ *such that* $p \cdot g = q$.

Example 2.2. *Let* V *be a real* n*-dimensional vector space and let* $P := \{v = (v_1, \ldots, v_n) \in V^n \mid v \text{ is a basis of } V\}$. *Then* P *is a smooth manifold of dimension* n^2. *The group* $G = \mathrm{Gl}(n; \mathbb{R})$ *acts on* P *from the right by*

$$P \times G \ni (v, A) \mapsto v \cdot A = \left(\sum_{i=1}^n A_{i,1} v_i, \ldots, \sum_{i=1}^n A_{i,n} v_i\right) \in P\,.$$

Indeed, we have $v \cdot E_n = v$ *for all* $v \in P$ *and if* $v \in P$, $A, B \in \mathrm{Gl}(n; \mathbb{R})$ *then*

$$\begin{aligned}(v \cdot A) \cdot B &= \left(\sum_{i=1}^n A_{i,1} v_i, \ldots, \sum_{i=1}^n A_{i,n} v_i\right) \cdot B = \left(\sum_{j=1}^n B_{j,1} \sum_{i=1}^n A_{i,j} v_i, \ldots, \sum_{j=1}^n B_{j,n} \sum_{i=1}^n A_{i,j} v_i\right) \\ &= \left(\sum_{i,j=1}^n A_{i,j} B_{j,1} v_i, \ldots, \sum_{i,j=1}^n A_{i,j} B_{j,n} v_i\right) = v \cdot (A \cdot B)\,.\end{aligned}$$

The action is smooth since it is a polynomial in the entries of its arguments. Moreover, it is easy to see that the action is simply-transitive.

Definition 2.3. *Let* G *be a Lie group and* M *a smooth manifold.*

(i) A G*-principal fibre bundle over* M *is a triple* $(P, \pi_P; G)$ *consisting of a manifold* P*, a smooth map* $\pi_P : P \to M$ *and a right-action of* G *on* P *such that*
   (a) π_P *is surjective,*
   (b) the action of G *on* P *is free,*
   (c) $\pi_P(p) = \pi_P(q)$ *if and only if there exists* $g \in G$ *such that* $p \cdot g = q$,
   (d) for every $x \in M$ *there exists an open neighborhood* $U \subseteq M$ *containing* x *and a section of* P *on* U*, i.e., a smooth map* $s_U : U \to P$ *such that* $\pi_p \circ s_U = \mathrm{id}_U$.

(ii) Let $(P, \pi_P; G)$ *and* $(Q, \pi_Q; G)$ *be* G*-principal fibre bundles over* M. *A smooth map* $\Phi : P \to Q$ *is called* G*-principal fibre bundle morphism if*
   (a) $\pi_Q \circ \Phi = \pi_P$ *and*
   (b) Φ *is (*G*-)equivariant, i.e., we have* $\Phi(p \cdot g) = \Phi(p) \cdot g$ *for all* $p \in P$ *and* $g \in G$.

(iii) The G*-principal fibre bundles* P *and* Q *are isomorphic, denoted* $P \cong Q$*, if there exists a* G*-principal fibre bundle isomorphism, i.e., a bijective* G*-principal fibre bundle morphism* $\Phi : P \to Q$.

Remark 2.4. *(i) By Definition 2.3(i)(b) and (c) G acts simply-transitively on every fibre $P_x := \pi_P^{-1}(x)$ of P over M.*
(ii) If there is no danger of confusion we will refer to the total space P of a G-principal fibre bundle $(P, \pi_P; G)$ as the principal fibre bundle.

Example 2.5. *Let M be a smooth manifold and G a Lie group. Define the manifold $P := M \times G$ with $\pi_P : P \ni (x, p) \mapsto x \in M$ and the G-action on P by multiplication of G from the right on the second factor. Then $(P, \pi_P; G)$ is a G-principal fibre bundle called the trivial G-principal fibre bundle over M.*

Proposition 2.6. *Let $(P, \pi_P; G)$ be a G-principal fibre bundle over M. Then $(P, \pi_P; G)$ is isomorphic to the trivial principal fibre bundle over M if and only if there exists a global section of P, i.e., a smooth map $s : M \to P$ such that $\pi_P \circ s = \mathrm{id}_M$.*

Proof. Let Q be the trivial G-principal fibre bundle over M. We can always define a global section over Q by

$$\begin{aligned} t : M &\to Q = M \times G \\ x &\mapsto (x, e) . \end{aligned}$$

If $\Phi : Q \to P$ is a G-principal fibre bundle isomorphism then $s := \Phi \circ t$ is a global section of P.

Now suppose that P admits a global section $s : M \to P$. Define a map

$$\begin{aligned} \Phi : Q = M \times G &\to P \\ (x, g) &\mapsto s(x) \cdot g . \end{aligned}$$

Then Φ is a G-principal fibre bundle isomorphism. □

Remark 2.7. *Not every G-principal fibre bundle P over M is isomorphic to the trivial principal fibre bundle. However, every such P is locally isomorphic to $M \times G$ in the following sense. For every $x \in M$ there exists an open neighborhood $U \subseteq M$ of x such that $\pi_P^{-1}(U) \cong U \times G$. Indeed, if $x \in M$ then by Definition 2.3(i)(d) there exists such a U and a section $s_U : U \to P$. The map*

$$\begin{aligned} \Phi : U \times G &\to \pi_P^{-1}(U) \\ (x, g) &\mapsto s_U(x) \cdot g \end{aligned}$$

is a G-principal fibre bundle isomorphism between $U \times G$ and $\pi_P^{-1}(U)$.

Example 2.8. *Let M be a smooth n-dimensional manifold. For $x \in M$ define*

$$\mathrm{Gl}(M)_x := \{v_x = (v_1, \ldots, v_n) \mid v_x \text{ is a basis of } T_x M\}$$

and let

$$\mathrm{Gl}(M) := \bigcup_{x \in M} \mathrm{Gl}(M)_x .$$

Define the projection via

$$\begin{aligned} \pi := \pi_{\mathrm{Gl}(M)} : \mathrm{Gl}(M) &\to M \\ v_x &\mapsto x . \end{aligned}$$

Note that if $(U, \varphi = (x^1, \ldots, x^n))$ is a coordinate chart of M, then for every $x \in U$ the associated frame $s_U(x) := (\partial_1(x), \ldots, \partial_n(x)) \in \mathrm{Gl}(M)_x$. The set $\mathrm{Gl}(M)$ has a unique structure as a smooth manifold if one requires that all such coordinate frames are smooth. This then turns $\pi_{\mathrm{Gl}(M)} : \mathrm{Gl}(M) \to M$ into a smooth map.

There is a $G = \mathrm{Gl}(n; \mathbb{R})$-right-action of $\mathrm{Gl}(n; \mathbb{R})$ on $\mathrm{Gl}(M)_x$ as defined in Example 2.2. This action induces a right-action of $\mathrm{Gl}(n; \mathbb{R})$ on $\mathrm{Gl}(M)$:

$$\begin{aligned} &\mathrm{Gl}(M) \times \mathrm{Gl}(n; \mathbb{R}) \to \mathrm{Gl}(M) \\ &(v_x = (v_1, \ldots, v_n), A) \mapsto v_x \cdot A = \left(\sum_{i=1}^n A_{i,1} v_i, \ldots, \sum_{i=1}^n A_{i,n} v_i \right) . \end{aligned} \tag{2.1}$$

The principal fibre bundle $(\mathrm{Gl}(M), \pi_{\mathrm{Gl}(M)}; \mathrm{Gl}(n; \mathbb{R}))$ is called the frame bundle of M.

Every additional structure on the manifold M defines a subbundle of $\mathrm{Gl}(M)$.

Example 2.9. *Let M again be a smooth n-dimensional manifold.*

(i) *Assume that M is oriented. Let $G = \mathrm{Gl}^+(n;\mathbb{R}) = \{A \in \mathrm{Gl}(n;\mathbb{R}) \mid \det A > 0\}$ and define*

$$\mathrm{Gl}^+(M) := \{v_x \in \mathrm{Gl}(M)_x \mid v_x \text{ is a positively oriented basis of } T_xM, x \in M\} .$$

We define a $\mathrm{Gl}^+(n;\mathbb{R})$ right-action on $\mathrm{Gl}^+(M)$ as the restriction of the $\mathrm{Gl}(n;\mathbb{R})$-action on $\mathrm{Gl}(M)$. With $\pi_{\mathrm{Gl}^+(M)} = \pi_{\mathrm{Gl}(M)|\mathrm{Gl}^+(M)}$, the tuple $(\mathrm{Gl}^+(M), \pi_{\mathrm{Gl}^+(M)}; \mathrm{Gl}^+(n;\mathbb{R}))$ is then a $\mathrm{Gl}^+(n;\mathbb{R})$-principal fibre bundle called the bundle of positively oriented frames .

(ii) *Let g be a Riemannian metric on M. Define*

$$\mathrm{O}(M) := \mathrm{O}(M,g) := \{v_x \in \mathrm{Gl}(M)_x \mid v_x \text{ is an orthonormal basis of } (T_xM, g_x)\} .$$

Analogously to before, we let $\pi_{\mathrm{O}(M)} : \mathrm{O}(M) \ni v_x \mapsto x \in M$ and define an $\mathrm{O}(n)$-right-action on $\mathrm{O}(M)$ by restricting the $\mathrm{Gl}(n;\mathbb{R})$-action on $\mathrm{Gl}(M)$. Then the $\mathrm{O}(n)$-principal fibre bundle $(\mathrm{O}(M), \pi_{\mathrm{O}(M)}; \mathrm{O}(n))$ is called the bundle of orthonormal frames of M .

(iii) *Combining the previous two examples leads us to the $\mathrm{SO}(n)$-principal fibre bundle of positively oriented orthonormal frames of M . That is, assume M is oriented and let g be a Riemannian metric on M. Define*

$$\mathrm{SO}(M) := \mathrm{SO}(M,g) := \{v_x \in \mathrm{Gl}(M)_x \mid v_x \text{ is a positively oriented orthonormal basis of } (T_xM, g_x)\}$$

and $\pi_{\mathrm{SO}(M)} : \mathrm{SO}(M) \ni v_x \mapsto x \in M$. Formula (2.1) defines an $\mathrm{SO}(n)$-right-action on $\mathrm{SO}(M)$ turning $(\mathrm{SO}(M), \pi_{\mathrm{SO}(M)}; \mathrm{SO}(n))$ into a principal fibre bundle.

A generalization of the notion of G-principal fibre bundle morphism is the following.

Definition 2.10. *Let $(P, \pi_P; G)$ be a G-principal fibre bundle over M and $f : H \to G$ a Lie group homomorphism. An f-reduction of P is a pair (Q, Φ) consisting of an H-principal fibre bundle $(Q, \pi_Q; H)$ over M and a smooth map $\Phi : Q \to P$ such that*

(i) $\pi_P \circ \Phi = \pi_Q$ *and*
(ii) $\Phi(q \cdot h) = \Phi(q) \cdot f(h)$ *for all $q \in Q$, $h \in H$.*

Properties (i) and (ii) can be summarized by saying that the diagram

$$\begin{tikzcd} Q \times H \arrow[r, "\cdot"] \arrow[d, "\Phi \times f"] & Q \arrow[d, "\Phi"] \arrow[dr, "\pi_Q"] & \\ P \times G \arrow[r, "\cdot"] & P \arrow[r, "\pi_P"] & M \end{tikzcd}$$

is commutative.

If we are in the situation that $H \subseteq G$ is a Lie subgroup and $f = \iota : H \hookrightarrow G$ is the inclusion, then we also call any f-reduction (Q, f) an H-reduction of P or a reduction of P to H .

Example 2.11. *Any of the principal fibre bundles from Example 2.9 together with the inclusion $\iota : H \to \mathrm{Gl}(n;\mathbb{R})$, $H = \mathrm{Gl}^+(n;\mathbb{R}), \mathrm{O}(n), \mathrm{SO}(n)$, is an H-reduction of the frame bundle $\mathrm{Gl}(M)$.*

Definition 2.12. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and M a smooth manifold.*

(i) *A $\mathbb{K}$-vector bundle of rank $k < \infty$ over M is a triple $(E, \pi_E; V)$ consisting of a smooth manifold E, a smooth map $\pi_E : E \to M$ and a k-dimensional $\mathbb{K}$-vector space V such that*
   (a) *π_E is surjective,*
   (b) *$E_x := \pi_E(x)^{-1}$ is $\mathbb{K}$-linearly isomorphic to V for all $x \in M$ and*
   (c) *for all $x \in M$ there exists an open neighborhood $U \subseteq M$ of x and k pointwise linearly independent local sections of E over U , i.e., there exist k smooth maps $s_1, \dots, s_k : U \to E$ such that*
      (1) *$\pi_E \circ s_j = \mathrm{id}_U$ for all $j = 1, \dots, k$ and*
      (2) *$(s_1(y), \dots, s_k(y))$ is a basis of E_y for all $y \in U$.*
      In case $\mathbb{K} = \mathbb{R}$ we call E a real vector bundle and in case $\mathbb{K} = \mathbb{C}$ a complex vector bundle .

(ii) *We denote the space of local sections of E over an open set $U \subseteq M$ by $\Gamma(U, E)$, i.e.,*

$$\Gamma(U, E) = \{s : U \to E \mid s \text{ is smooth and } \pi_E \circ s = \mathrm{id}_U\} .$$

In the particular case $U = M$ we call the elements of $\Gamma(U, E)$ just sections of E or sometimes global sections of E .

(iii) *For sections $s_1, \dots, s_k : U \to E$ as in (i)(c) we call the smooth map $s = (s_1, \dots, s_k) : U \to E^k$ a (local) frame for E. In case $U = M$, we call s a global frame for E .*

(iv) *Let E, F be two $\mathbb{K}$-vector bundles over M. A smooth map $\Phi : E \to F$ is a vector bundle homomorphism if*
  (a) $\pi_F \circ \Phi = \pi_E$ *and*
  (b) $\Phi_{|E_x} : E_x \to F_x$ *is $\mathbb{K}$-linear for all $x \in M$.*
  We call Φ a vector bundle isomorphism if it is invertible and then we call E and F isomorphic .

Example 2.13. *Let M be a smooth manifold.*

(i) *Let V be a k-dimensional $\mathbb{K}$-vector space. Define $E := M \times V$ and $\pi_E : E = M \times V \ni (x, v) \mapsto x \in M$. If we define*

$$(x, v) + (x, w) := (x, v + w)\,,$$
$$\lambda \cdot (x, v) := (x, \lambda \cdot v)$$

for all $x \in M$, $v, w \in V$ and $\lambda \in \mathbb{K}$, then $(E, \pi_E; V)$ is a rank k vector bundle over M. We call E the trivial vector bundle with fibre V over M , or simply trivial .

The sections $\Gamma(M, E)$ are smooth maps $s : M \to E = M \times V$ satisfying $\pi_E \circ s(x) = x$, hence they are of the form $s(x) = (x, v(x))$ for some $v \in C^\infty(M, V)$.

(ii) *The tanget bundle TM of M is a real vector bundles of rank $k = \dim M$ over M. The sections $\Gamma(M, TM)$ of TM are precisely the smooth vector fields $\mathcal{V}(M)$.*

Remark 2.14. *Note that the space of sections $\Gamma(M, E)$ of the $\mathbb{K}$-vector bundle E over M is a modul over the ring $C^\infty(M; \mathbb{K})$ of smooth $\mathbb{K}$-valued functions on M. Here, the sum of two sections and the product of a smooth function and a section of E are defined pointwise, i.e., for $f \in C^\infty(M, \mathbb{K})$ and $s, t \in \Gamma(M, E)$ the sections $s + t, fs \in \Gamma(M, E)$ are defined by*

$$(s + t)(x) := s(x) + t(x) \in E_x\,,$$
$$(fs)(x) := f(x)s(x) \in E_x$$

for all $x \in M$.

In linear algebra we learn how to construct new vector spaces out of given ones, e.g., the dual vector space, the direct sum or tensor product of two vector spaces. These constructions carry directly over to vector bundles.

Definition 2.15. (i) *Let $(E, \pi_E; V)$ and $(F, \pi_F; W)$ be two $\mathbb{K}$-vector bundles of rank k and l, respectively, over M. The Whitney-Sum of E and F is the $\mathbb{K}$-vector bundle $(E \oplus F, \pi_{E\oplus F}; V \oplus W)$, where*

$$E \oplus F := \bigcup_{x \in M} E_x \oplus F_x$$

and

$$\pi_{E\oplus F} : E \oplus F \ni (e_x, f_x) \mapsto x \in M\,.$$

If $x \in M$ and $U, V \subseteq M$ are neighborhoods of x such that there are local frames $s = (s_1, \ldots, s_k) : U \to E^k$ and $t = (t_1, \ldots, t_l) : V \to F^l$, then the $k + l$ maps

$$s_{1|W}, \ldots, s_{k|W} : W \to E \subseteq E \oplus F\,, \qquad t_{1|W}, \ldots, t_{l|W} : W \to F \subseteq E \oplus F$$

where $W := U \cap V$, are a pointwise linearly independent. The requirement that all these collections of maps are smooth equips $E \oplus F$ with a topology and a smooth structure, which then turns $\pi_{E\oplus F}$ into a smooth map.

(ii) *As above, let $(E, \pi_E; V)$ and $(F, \pi_F; W)$ be two $\mathbb{K}$-vector bundles of rank k and l, respectively, over M. We consider the set*

$$E \otimes F := \bigcup_{x \in M} E_x \otimes_{\mathbb{K}} F_x$$

with the projection

$$\pi_{E\otimes F} : E \otimes F \ni \sum_{i,j} e_x^i \otimes f_x^j \mapsto x \in M\,.$$

For local frames s of E and t of F as above, the $k \cdot l$ maps

$$u_{i,j} : W \to E \otimes F \quad i = 1, \ldots, k \text{ and } j = 1, \ldots, l$$

with

$$u_{i,j}(y) = s_i(y) \otimes t_j(y) \quad \text{for all} \quad y \in W\,,$$

are pointwise linearly independent. The requirement that all such maps constructed out of local frames of E and F are smooth turns $E \otimes F$ *into a smooth manifold and* $\pi_{E \otimes F}$ *into a smooth map. The vector bundle* $(E \otimes F, \pi_{E\otimes F}; V \otimes W)$ *is called the tensor product of E and F .*

(iii) *Let* $(E, \pi_E; V)$ *be a* $\mathbb{K}$*-vector bundle. We consider the set*

$$E^* := \bigcup_{x \in M} E_x^*$$

and the projection

$$\pi_{E*} : E^* \ni \alpha_x \mapsto x \in M .$$

If $s = (s_1, \dots, s_k) : U \to E^k$ *is a local frame of E, then we define the dual frame* $\varphi = (\varphi_1, \dots, \varphi_k) : U \to (E^*)^k$ *by requiring that*

$$(\varphi_1(x), \dots, \varphi_k(x))$$

is the basis of E_x^* *dual to the basis* $(s_1(x), \dots, s_k(x))$ *of* E_x*, for all* $x \in U$*. That is,* $\varphi_i(x)(s_j(x)) = \delta_{i,j}$ *for all* $x \in U$*. The requirement that all such dual frames are smooth turns* E^* *into a smooth manifold and* π_{E*} *into a smooth map. The vector bundle* $(E^*, \pi_{E*}; V^*)$ *is the dual vector bundle of E .*

(iv) *Let* $(E, \pi_E; V)$ *be a complex vector bundle over M and let* $\overline{V}$ *be the complex conjugate vector space. That is,* $\overline{V}$ *is the abelian group V together with the scalar multiplication* $\mathbb{C} \times V \ni (z, v) \mapsto \overline{z} \cdot v \in V$*. We consider the set*

$$\overline{E} := \bigcup_{x \in M} \overline{E_x}$$

with projection

$$\pi_{\overline{E}} : \overline{E} \ni e_x \mapsto x \in M .$$

Any local frame $s = (s_1, \dots, s_k) : U \to E^k$ *defines a a local frame* $s : U \to \overline{E}^k$*. Thus,* $\overline{E}$ *directly inherits the topology and smooth structure from E. The vector bundle* $(\overline{E}, \pi_{\overline{E}}; \overline{V})$ *is the complex conjugate vector bundle of E .*

In case $(E, \pi_E; V)$ *is a real vector bundle we define* $(\overline{E}, \pi_{\overline{E}}; \overline{V}) := (E, \pi_E; V)$.

(v) *There exist many more constructions like* $\mathrm{Hom}(E, F)$, $\Lambda^l E, \dots$

Remark 2.16. (i) *In case of the tangent bundle TM of a smooth manifold M, the dual bundle* TM^**, called cotangent bundle, is denoted* T^*M*. Note also that in case of the tangent and cotangent bundle we denote the individual fibres by* T_xM *and* T_x^*M *instead of* TM_x *and* T^*M_x*, respectively.*

(ii) *Note that the above operations* $\oplus, \otimes, ^*, \dots$ *induce associated operations on the corresponding sections. For example, if* $s \in \Gamma(M, E)$ *and* $t \in \Gamma(M, F)$*, then* $s \otimes t \in \Gamma(M, E \otimes F)$ *is the section defined by* $(s \otimes t)(x) := s(x) \otimes t(x)$*.*

Example 2.17. *We consider the real vector bundle* $T^*M \otimes T^*M$*. An element* $b \in (T^*M \otimes T^*M)_x = T_x^*M \otimes T_x^*M$ *(*$x \in M$*) can be thought of as a bilinear form, i.e., given* $v, w \in T_xM$ *we have* $b(v, w) \in \mathbb{R}$*. As usual, we call b symmetric if* $b(v, w) = b(w, v)$ *for all* $v, w \in T_xM$ *and positive definite if* $b(v, v) > 0$ *for all* $v \in T_xM\backslash\{0\}$*. A Riemannian metric g on M is nothing but an element of* $\Gamma(M, T^*M \otimes T^*M)$ *that is pointwise symmetric and positive definite. In other words, g is a pointwise inner product depending smoothly on the basepoint.*

More generally than the example of a Riemannian metric, we have the notion of a bundle metric.

Definition 2.18. *Let* $(E, \pi_E; V)$ *be a real or complex vector bundle over M. A bundle metric on E is a section* $\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes \overline{E}^*)$ *which is pointwise an inner product, that is, pointwise symmetric and positive definite (*$\mathbb{K} = \mathbb{R}$*) respectively hermitian and positive definite (*$\mathbb{K} = \mathbb{C}$*).*

Remark 2.19. *Just as for Riemannian metrics, a simple argument using a partition of unity shows that any vector bundle carries a bundle metric.*

So far, we have introduced two different types of fibre bundles, namely principal fibre bundles and vector bundles. The next definition connects these two seamingly different worlds.

Definition 2.20. *Let M be a smooth manifold,* $(P, \pi_P; G)$ *a G-principal fibre bundle over M and* $\rho : G \to \mathrm{Gl}(V)$ *a real resp. complex representation of G on V. Define the set*

$$E := P \times_\rho V := P \times_{(G,\rho)} V := G \times V / \sim$$

where the equivalence relation $\sim$ *is given by*

$$(p, v) \sim (p \cdot g, \rho(g^{-1})(v)) \quad \text{for all } g \in G ,$$

the projection $\pi_E : E \ni [p, v] \mapsto \pi_P(p) \in M$, and on each fibre $E_x = P_x \times_{(G,\rho)} V$ the vector space structure

$$\mu[p, v] + \nu[p, w] := [p, \mu v + \nu w] \quad \text{for all} \quad p \in P, v, w \in V, \mu, \nu \in \mathbb{K}\,.$$

We equip E with a topology and smooth structure by requiring that if $s : U \to P$ is a local section of P and $v \in C^\infty(U, V)$, then $U \ni x \mapsto [s(x), v(x)] \in E$ is smooth. The real (V real) resp. complex (V complex) vector bundle $(E, \pi_E; V)$ is the vector bundle associated with P and ρ .

Remark 2.21. *With respect to the construction in the last definition, the operations $\oplus, \otimes, {}^*, \mathsf{Hom}, \ldots$ on vector bundles correspond exactly to the operations denoted by the same symbols on representations.*

Example 2.22. *Let M be a smooth manifold, $\mathrm{Gl}(M)$ the frame bundle of M and $\rho : \mathrm{Gl}(n; \mathbb{R}) \to \mathrm{Gl}(\mathbb{R}^n)$ the standard representation. Then*

$$\begin{aligned} \Phi : \mathrm{Gl}(M) \times_\rho \mathbb{R}^n &\to TM \\ [(s_1, \ldots, s_n), (x_1, \ldots, x_n)^t] &\mapsto \sum_{i=1}^n x_i s_i \end{aligned}$$

is a vector bundle isomorphism. If $\rho^* : \mathrm{Gl}(n; \mathbb{R}) \to \mathrm{Gl}((\mathbb{R}^n)^*)$ is the representation dual to ρ, i.e., $\rho^*(g)(l)(x) = l(\rho(g^{-1})x)$ for all $l \in (\mathbb{R}^n)^*$ and $x \in \mathbb{R}^n$, then

$$\begin{aligned} \Psi : \mathrm{Gl}(M) \times_{\rho^*} (\mathbb{R}^n)^* &\to TM \\ [(s_1, \ldots, s_n), (y_1, \ldots, y_n)] &\mapsto \sum_{i=1}^n y_i \sigma_i\,, \end{aligned}$$

where $(\sigma_1, \ldots, \sigma_n)$ is the basis dual to $(s_1, \ldots, s_n)$, is a vector bundle isomorphism.

Proposition 2.23. *Let M be a smooth manifold, $(P, \pi_P; G)$ a G-principal fibre bundle over M and $\rho : G \to \mathrm{Gl}(V)$ a representation. If there exists a G-invariant inner product $\langle \cdot, \cdot \rangle$ on V then on the vector bundle $E = P \times_\rho V$ associated with P and ρ there exists a bundle metric given by*

$$\langle e, f \rangle_{E_x} := \langle v, w \rangle\,,$$

where $e = [p, v]$ and $f = [p, w]$ for some $p \in P_x$.

Proof. We have to show that the bundle metric is well-defined, i.e., independent of the chosen representatives. Let $q \in P_x$ and let $g \in G$ be the unique element such that $q = p \cdot g$. Then we have by definition $e = [p, v] = [p \cdot g, \rho(g^{-1})(v)] = [q, \rho(g^{-1})(v)]$ and $f = [p, w] = [p \cdot g, \rho(g^{-1})(w)] = [q, \rho(g^{-1})(w)]$. Since the inner product on V is G-invariant, we have $\langle v, w \rangle = \langle \rho(g^{-1})(v), \rho(g^{-1})(w) \rangle$. Hence, the bundle metric is well-defined. □

Definition 2.24. *Let $(E, \pi_E; V)$ be a $\mathbb{K}$-vector bundle over M.*

(i) A linear map

$$\nabla : \Gamma(M, E) \to \Gamma(T^*M \otimes E)$$

is called covariant derivative on E if

$$\nabla(fs) = \mathrm{d}f \otimes s + f \cdot \nabla s \quad \text{for all} \quad f \in C^\infty(M, \mathbb{K}), s \in \Gamma(M, E)\,.$$

If $s \in \Gamma(M, E)$ and $X \in \mathcal{V}(M)$, then the section $\nabla_X s := \nabla s(X) \in \Gamma(E)$ is called covariant derivative of s in direction X.

(ii) If E comes with a bundle metric, a covariant derivative ∇ in E is called metric if

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$$

for all $X \in \mathcal{V}(M), s, t \in \Gamma(M, E)$. Here, $\langle s, t \rangle \in C^\infty(M)$ is the function $\langle s, t \rangle(x) := \langle s(x), t(x) \rangle_{E_x}$.

Example 2.25. *The Levi-Civita connection of a Riemannian manifold $(M, \langle \cdot, \cdot \rangle = g)$ is the unique covariant derivative ∇^{LC} on $E = TM$ given by the Koszul formula*

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(X\langle Y, Z \rangle + Y\langle Z, Y \rangle - Z\langle Y, X \rangle + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle \right)\,.$$

The Levi-Civita connection is metric and, moreover, torsionfree, i.e., $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$.

Note that the torsion tensor T can in general only be defined on the tangent bundle and not on an arbitrary vector bundle E.

3. Spin Geometry

Definition 3.1. *Let (M, g) be an oriented Riemannian manifold.*

(i) A Spin-structure on M is a pair (P, π) consisting of a $\mathrm{Spin}(n)$-principal fibre bundle $(P, \pi_P; \mathrm{Spin}(n))$ over M and a smooth map $\pi : P \to \mathrm{SO}(M, g)$ such that

(a) $\pi_{\mathrm{SO}(M)} \circ \pi = \pi_P$ and

(b) $\pi(p \cdot g) = \pi(p) \cdot \lambda(g)$ for all $p \in P$ and $g \in \mathrm{Spin}(n)$ with $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ the Lie group homomorphism from Section 1.3.

In other words, a Spin-structure on M is a λ-reduction of the bundle $\mathrm{SO}(M)$ of oriented orthonormal frames of M. We can summarize properties (a) and (b) by saying that the diagram

$$\begin{tikzcd}
P \times \mathrm{Spin}(n) \arrow[r, "\cdot"] \arrow[d, "\pi\times\lambda"] & P \arrow[d, "\pi"] \arrow[dr, "\pi_P"] & \\
\mathrm{SO}(M,g) \times \mathrm{SO}(n) \arrow[r, "\cdot"] & \mathrm{SO}(M) \arrow[r, "\pi_{\mathrm{SO}(n)}"] & M
\end{tikzcd}$$

is commutative.

(ii) Two Spin-structures (P_1, π_1) and (P_2, π_2) on M are called equivalent if there exists a $\mathrm{Spin}(n)$-principal fibre bundle isomorphism $\Phi : P_1 \to P_2$ such that $\pi_1 = \pi_2 \circ \Phi$.

(iii) If there exists a Spin-structure on a Riemannian manifold (M, g), we call M spin .

Remark 3.2. *Note that two equivalent Spin-structures (P_1, π_1) and (P_2, π_2) on M provide isomorphic $\mathrm{Spin}(n)$-principal fibre bundles P_1 and P_2. However, the converse is not true. There do exist oriented Riemannian manifolds (M, g) having two inequivalent Spin-structures (P_1, π_1) and (P_2, π_2) such that P_1 and P_2 are isomorphic as abstract $\mathrm{Spin}(n)$-principal fibre bundles over M.*

Example 3.3. *Let $M = \mathbb{R}^n$. By identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$, we can equip $\mathbb{R}^n$ with the Riemannian metric g given by the Euclidean inner product,*

$$g_x(v, w) := \langle v, w\rangle \quad \text{for all} \quad x \in \mathbb{R}^n, v, w \in T_x\mathbb{R}^n = \mathbb{R}^n \,,$$

and its standard orientation given by requiring that the canonical basis $(e_1, \ldots, e_n)$ of $T_x\mathbb{R}^n = \mathbb{R}^n$ is positively oriented.

The bundle $\mathrm{SO}(\mathbb{R}^n, g)$ of oriented orthonormal frames is trivial, i.e., is given by

$$\mathrm{SO}(\mathbb{R}^n, g) = \mathbb{R}^n \times \mathrm{SO}(n) \,,$$

where we have identified an OONB $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$ with the matrix $A \in \mathrm{SO}(n)$ whose i-th column is v_i. A Spin-structure for $(\mathbb{R}^n, g)$ is now given by (P, π) with

$$P = \mathbb{R}^n \times \mathrm{Spin}(n)$$

and

$$\begin{aligned} \pi : P = \mathbb{R}^n \times \mathrm{Spin}(n) &\to \mathbb{R}^n \times \mathrm{SO}(n) = \mathrm{SO}(\mathbb{R}^n, g) \\ (x, g) &\mapsto (x, \lambda(g)) \,. \end{aligned}$$

Example 3.4. *We consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ with its round standard metric g, i.e.,*

$$g_x(v, w) := \langle v, w\rangle \quad \text{for all} \quad x \in S^n, v, w \in T_xS^n \subseteq T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \,,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. By our identification $T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$, we have

$$T_xS^n = x^\perp = \{v \in \mathbb{R}^{n+1} \,|\, \langle v, x\rangle = 0\} \,.$$

The orientation we endow S^n with is defined by requiring any basis $(v_1, \ldots, v_n)$ of T_xS^n to be oriented if and only if $(v_1, \ldots, v_n, x)$ is an oriented basis of $\mathbb{R}^{n+1}$. It follows that for any positively oriented orthonormal basis $(v_1, \ldots, v_n)$ of T_xS^n, $(v_1, \ldots, v_n, x)$ is an oriented orthonormal basis of $\mathbb{R}^{n+1}$. Thus, the bundle $\mathrm{SO}(S^n)$ is given by

$$\mathrm{SO}(S^n) = \mathrm{SO}(n + 1) \,,$$

where we have identified the OONB $(v_1, \ldots, v_n, x)$ of $\mathbb{R}^{n+1}$ with the matrix A in $\mathrm{SO}(n + 1)$ having $v_1, \ldots, v_n, x$ as columns, with projection

$$\begin{aligned} \pi_{\mathrm{SO}(S^n)} : \mathrm{SO}(S^n) = \mathrm{SO}(n + 1) &\to S^n \\ (v_1, \ldots, v_n, x) = A &\mapsto x = A \cdot e_{n+1} \,. \end{aligned}$$

The right-action of $\mathrm{SO}(n)$ *on* $\mathrm{SO}(S^n) = \mathrm{SO}(n+1)$ *is given by the right-multiplication of* $\mathrm{SO}(n+1)$ *on itself precomposed with the inclusion*

$$\begin{aligned} \iota : \mathrm{SO}(n) &\to \mathrm{SO}(n+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Associated with the inclusion ι *is an inclusion* $\tilde{\iota} : \mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$*, which can be constructed as follows. The inclusion* $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^{n+1}$ *induces an inclusion* $\mathcal{C}\ell_n \hookrightarrow \mathcal{C}\ell_{n+1}$ *(the image of which is the algebra generated by* $e_1, \dots, e_n$*), which, by restriction, induces an inclusion* $\tilde{\iota} : \mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$*. It follows from the construction of the map* λ *from Section 1.3 that* $\lambda_{n+1}(\tilde{\iota}(g)) = \iota(\lambda_n(g))$ *for all* $g \in \mathrm{Spin}(n)$*.*

To construct our Spin-structure for S^n *we set* $P := \mathrm{Spin}(n+1)$*. The right-action of* $\mathrm{Spin}(n)$ *on* P *is given by right-multiplication of* $\mathrm{Spin}(n+1)$ *on itself precomposed with the inclusion* $\tilde{\iota}$*. We set* $\pi := \lambda_{n+1} : P = \mathrm{Spin}(n+1) \to \mathrm{SO}(n+1) = \mathrm{SO}(S^n)$ *and define the projection* $\pi_P : P \to S^n$ *which makes* P *into a principal fibre bundle over* S^n *by* $\pi_P := \pi_{\mathrm{SO}(S^n)} \circ \lambda_{n+1}$*. Now* (P, π) *is a Spin-structure for* S^n*. We summarize the situation in two commutative diagrams:*

$$\begin{tikzcd} P = \mathrm{Spin}(n+1) \arrow[d, "\lambda_{n+1}"] \arrow[dd, bend right=60, swap, "\pi_P = \pi_{\mathrm{SO}(S^n)} \circ \lambda_{n+1}"] \\ \mathrm{SO}(S^n) = \mathrm{SO}(n+1) \arrow[d, "\pi_{\mathrm{SO}(S^n)}"] \\ S^n \end{tikzcd} \qquad \begin{tikzcd} P \times \mathrm{Spin}(n) \arrow[r, "\cdot \circ (\mathrm{id} \times \tilde{\iota})"] \arrow[d, "\lambda_{n+1} \times \lambda_n"] & P \arrow[d, "\lambda_{n+1}"] \\ \mathrm{SO}(S^n) \times \mathrm{SO}(n) \arrow[r, "\cdot \circ (\mathrm{id} \times \iota)"] & \mathrm{SO}(S^n) \end{tikzcd}$$

Example 3.5. *Let* $M = S^1 \cong [0, 2\pi]/\{0, 2\pi\}$ *with the metric it inherits from its embedding into* $\mathbb{C} \cong \mathbb{R}^2$ *and the counterclockwise orientation. Since in dimension 1 there is only one positively oriented unit-vector in each tangent space, we see that* $\mathrm{SO}(S^1) \cong S^1$*. Note that* $\mathrm{SO}(1) = \{1\}$ *and* $\mathrm{Spin}(1) = \{\pm 1\} = \mathbb{Z}_2$*. The first Spin-structure we define is* $P_1 := S^1 \times \mathbb{Z}_2$ *with the obvious projections and right-action of* $\mathbb{Z}_2$*. We call* P_1 *the trivial Spin-structure on* S^1 *. There is a second Spin-structure on* S^1*. Define* $P_2 := [0, 2\pi] \times \mathbb{Z}_2 / \sim$ *where* $(0, \pm 1) \sim (2\pi, \mp 1)$ *with projection onto* S^1 $\pi_{P_2}([x, g]) = x$*. We call* P_2 *the nontrivial Spin-structure on* S^1 *. The two Spin-structures are inequivalent.*

Remark 3.6. *Not every Riemannian manifold allows a Spin-structure. Examples are the even-dimensional real projective spaces* $\mathbb{RP}^{2m}$*, which are not orientable and so, in particular, not spin. Orientable examples, which are not spin, are the even-dimensional complex projective spaces* $\mathbb{CP}^{2m}$*.*

It is remarkable that, although the definition of a Spin-structure explicitely references the Riemannian metric, the existence of a Spin-structure and the number of inequivalent Spin-structures are independent of the metric in the sense that if a manifold M *admits a Spin-structure for one Riemannian metric* g*, then it does so for every other Riemannian metric and the number of inequivalent Spin-structures is constant when viewed as a function of the metric. In fact, even more is true: a manifold is spin if and only if its second Stiefel-Whitney class vanishes and then it admits as many inequivalent Spin-structures as there are elements in* $H^1(M; \mathbb{Z}_2)$*. In particular, being spin is a* topological *invariant.*

For the next definition recall the associated vector bundle construction from Definition 2.20.

Definition 3.7. *(i) Let* (M, g) *be an oriented* n*-dimensional Riemannian manifold with Spin-structure* (P, π)*. Let* $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$ *be the fundamental Spin-representation. The complex vector bundle*

$$\Sigma M := P \times_{\kappa_n} \Sigma_n$$

is called the spinor bundle of (M, g) *and the Spin-structure* (P, π) *.*

(ii) A section $s \in \Gamma(M, \Sigma M)$ *is called a spinor field or, sloppily, a spinor .*

Remark 3.8. *(i) The spinor bundle* ΣM *has rank* $\dim \Sigma_n = 2^{\lfloor \frac{n}{2} \rfloor}$*. Moreover, since* κ_n *is a unitary representation it comes equipped with a canonical bundle metric as described in Proposition 2.23.*

(ii) Recall that in case $n = 2m$ *the fundamental spin representation splits into the direct sum* $\kappa_{2m} = \kappa_{2m}^+ \oplus \kappa_{2m}^-$ *of the positive respectively negative half-spin representations* $\kappa_{2m}^\pm : \mathrm{Spin}(2m) \to \mathrm{U}(\Sigma_{2m}^\pm)$*. To this splitting corresponds a splitting of the spinor bundle (see Remark 2.21)*

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M,$$

where the vector bundles

$$\Sigma^{\pm} M := P \times_{\kappa^{\pm}} \Sigma_n^{\pm}$$

are called the bundles of positive respectively negative half-spinors . The sections $s \in \Gamma(M, \Sigma^{\pm} M)$ are called positive respectively negative half-spinors .

Remark 3.9. *While $\mathbb{R}^n$ is a real vector space, the space of spinors Σ_n is a complex space. We can view Σ_n a real vector space by restricting scalar multiplication to $\mathbb{R}$. This allows us to consider the (real) tensor product $\mathbb{R}^n \otimes \Sigma_n$. But note that $\mathbb{R}^n \otimes \Sigma_n$ carries a canonical structure as a complex vector space where scalar multiplication with complex numbers is given by multiplication on the second factor.*

The analogous statement applies to the real vector bundle TM, the complex vector bundle ΣM and their tensor product $TM \otimes \Sigma M$.

Definition 3.10. *Let (M, g) be an oriented Riemannian manifold with a Spin-structure (P, π) and let ΣM be the associated spinor bundle. A Clifford multiplication is a vector bundle homormorphism of complex vector bundles*

$$\begin{aligned} \mu : TM \otimes \Sigma M &\to \Sigma M \\ v \otimes \sigma &\mapsto v \cdot \sigma \end{aligned}$$

satisfying

$$v \cdot (w \cdot \sigma) + w \cdot (v \cdot \sigma) = -2g(v, w) \cdot \sigma \quad \text{for all} \quad x \in M, v, w \in T_x M, \sigma \in \Sigma M_x .$$

Proposition 3.11. *Let (M, g) be a Riemannian spin manifold with spin structure (P, π) and let ΣM be the associated spinor bundle.*

(i) If n is even there exists exactly one Clifford multiplication. If n is odd there exist exactly two Clifford multiplications which are the negative of each other. They can be distinguished by the action of the complex volume element, i.e., we have either

$$\omega_n^{\mathbb{C}} \cdot \sigma := i^{\lfloor (n+1)/2 \rfloor} e_1 \cdot (e_2 \cdot (\ldots (e_n \cdot \sigma))) = \sigma \qquad \text{for all} \quad x \in M, \sigma \in \Sigma M_x ,$$

or

$$\omega_n^{\mathbb{C}} \cdot \sigma = -\sigma \qquad \text{for all} \quad x \in M, \sigma \in \Sigma M_x ,$$

where $(e_1, \ldots, e_n)$ is an OONB of $T_x M$.

(ii) Any Clifford multiplication satisfies

$$\langle v \cdot \sigma, \tau \rangle = -\langle \sigma, v \cdot \tau \rangle \quad \text{for all} \quad x \in M, v \in T_x M, \sigma, \tau \in \Sigma M_x .$$

Proof. To proof (i), we first note that the tangent bundle TM is associated to the Spin-structure (P, π) via the representation $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. More precisely, the vector bundle homomorphism

$$\begin{aligned} P \times_{\lambda} \mathbb{R}^n &\to TM \\ [p, (x_1, \ldots, x_n)^t] &\mapsto \sum_{i=1}^n x_i \pi(p)_i \end{aligned}$$

is an isomorphism. Here, for $p \in P_x$ we have $\pi(p) = (\pi(p)_1, \ldots, \pi(p)_n) \in \mathrm{SO}(M)_x$. Alluding to Remark 2.21 again, it follows that the vector bundle $TM \otimes \Sigma M$ is associated to P and the representation $\lambda \otimes \kappa_n : \mathrm{Spin}(n) \to \mathrm{Gl}(\mathbb{R}^n \otimes \Sigma_n)$ through the isomorphism

$$\begin{aligned} P \times_{\lambda \otimes \kappa_n} (\mathbb{R}^n \otimes \Sigma_n) &\to TM \otimes \Sigma M \\ [p, x \otimes \sigma] &\mapsto \sum_{i=1}^n x_i \pi(p)_i \otimes [p, \sigma] . \end{aligned}$$

If $\widetilde{\mu} : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ is any Clifford multiplication as in Definition 1.52, we define the Clifford multiplication

$$\begin{aligned} \mu : TM \otimes \Sigma M \cong P \times_{\lambda \otimes \kappa_n} (\mathbb{R}^n \otimes \Sigma_n) &\to P \times_{\kappa_n} \Sigma_n \cong \Sigma M \\ [p, x \otimes \sigma] &\mapsto [p, \widetilde{\mu}(x \otimes \sigma)] = [p, x \cdot \sigma] . \end{aligned}$$

We have to check that μ is well-defined, i.e., is independent of the chosen representative. For this, let $p, q \in P_x$ and let $g \in \mathrm{Spin}(n)$ be the unique element such that $q = p \cdot g$. Then we have

$$[p, x \otimes \sigma] = [p \cdot g, (\lambda \otimes \kappa_n)(g^{-1})(x \otimes \sigma)] = [q, \lambda(g^{-1})(x) \otimes \kappa_n(g^{-1})(\sigma)]$$

and

$$[p, \widetilde{\mu}(x \otimes \sigma)] = [p \cdot g, \kappa_n(g^{-1})(\widetilde{\mu}(x \otimes \sigma))] = [q, \kappa_n(g^{-1})(\widetilde{\mu}(x \otimes \sigma))] .$$

From Proposition 1.56 we know that

$$\kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma)) = \widetilde{\mu}((\lambda\otimes\kappa_n)(g^{-1})(x\otimes\sigma)) = \widetilde{\mu}(\lambda(g^{-1})(x)\otimes\kappa_n(g^{-1})(\sigma))$$

so that

$$[p,\widetilde{\mu}(x\otimes\sigma)] = [q,\kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma))] = [q,\widetilde{\mu}(\lambda(g^{-1})(x)\otimes\kappa_n(g^{-1})(\sigma))]$$

as required.

All statements now follow from Proposition 1.53 and Corollary 1.54. □

Remark 3.12. *(i) In case the dimension n of M is odd, we will always fix the Clifford multiplication for which the complex volume element acts by $+\,\mathrm{id}_{\Sigma M}$.*

(ii) We extend the Clifford multiplication to vector and spinor fields, that is, for $X \in \mathcal{V}(M)$ and $\varphi \in \Gamma(M,\Sigma M)$ we let $X\cdot\varphi$ be the spinor field defined by

$$(X\cdot\varphi)_x := X_x\cdot\varphi(x) \quad \text{for all} \quad x\in M\,.$$

All relations holding pointwise then also hold as field equations, e.g., we have

$$X\cdot(Y\cdot\varphi)+Y\cdot(X\cdot\varphi) = -2g(X,Y)\cdot\varphi \quad \text{for all} \quad X,Y\in\mathcal{V}(M),\varphi\in\Gamma(M,\Sigma M)\,.$$

Theorem 3.13. *There exists a metric connection $\nabla = \nabla^{\Sigma} : \Gamma(M,\Sigma M)\to\Gamma(M,T^*M\otimes\Sigma M)$ on ΣM satisfying*

$$\nabla^{\Sigma}_X(Y\cdot\varphi) = \nabla_X Y\cdot\varphi + Y\cdot\nabla^{\Sigma}_X\varphi \quad \text{for all} \quad X,Y\in\mathcal{V}(M),\varphi\in\Gamma(M,\Sigma M)\,. \tag{3.1}$$

The connection ∇^{Σ} is called spinor connection or Levi-Civita connection .

Remark. *In fact, ∇^{Σ} is the unique metric connection satisfying* (3.1). *Unfortunately, we will have to content ourselves with the existence of ∇.*

Proof. Let (P,π) be our Spin-structure with which ΣM is associated.

Step 1: For any local section $s : M \supseteq U \to P$ let $(\mathbf{e}_1,\ldots,\mathbf{e}_n) := \pi\circ s : U \to \mathrm{SO}(M,g)$ be the projected local OONB. For any $\varphi\in\Gamma(U,\Sigma M)$, given by $\varphi = [s,v]$ for some $v\in C^\infty(U,\Sigma_n)$, define

$$\nabla^s_X\varphi = [s,X(v)] + \frac{1}{4}\sum_{i=1}^n \mathbf{e}_i\cdot\nabla^{\mathrm{LC}}_X\mathbf{e}_i\cdot\varphi \tag{3.2}$$

for any $X\in\mathcal{V}(M)$. Obviously, ∇^s is $\mathbb{C}$-linear with respect to φ, $C^\infty(U,\mathbb{C})$-linear w.r.t. X and satisfies the Leibniz rule.

Step 2: We show that (3.2) is independent of the section s. Let s,t be local sections of P, which are, without loss of generality, defined on the same open set $W\subseteq M$. We let $\sigma : W\to\mathrm{Spin}(n)$ be the unique smooth map which satisfies

$$t = s\cdot\sigma$$

and $v,w\in\mathbb{C}^\infty(W,\Sigma_n)$ such that $\varphi = [s,v] = [t,w]$. Then

$$[s,v] = [s\cdot\sigma,\kappa_n(\sigma^{-1})(v)] = [t,w]\,.$$

We consider the first term on the right-hand side of (3.2). We have

$$X(w) = X(\kappa_n(\sigma^{-1})(v)) = (\mathrm{d}(\kappa_n\circ\sigma^{-1})X)(v)+\kappa_n(\sigma^{-1})(X(v))\,.$$

Since $\mathrm{Spin}(n)\subseteq\mathcal{C}\ell_n^*$, we have $\mathrm{d}(L_g)X = g\cdot X$ respectively $\mathrm{d}(R_g)X = X\cdot g$ (cf. Example 1.14) and using Exercise 16 we see that

$$\begin{aligned}(\mathrm{d}(\kappa_n\circ\sigma^{-1})X)(v) &= (\mathrm{d}\kappa_n\circ\mathrm{d}\,\mathrm{inv}\circ\mathrm{d}\sigma X)(v) = -\kappa_n(\mathrm{d}(L_{\sigma^{-1}})\circ\mathrm{d}(R_{g^{-1}})\mathrm{d}\sigma X)(v)\\ &= -\kappa_n(\sigma^{-1}(\mathrm{d}\sigma X)\sigma^{-1})(v) = -\kappa_n(\sigma^{-1})\kappa_n(\mathrm{d}\sigma X\cdot\sigma^{-1})(v)\,,\end{aligned}$$

so that

$$\begin{aligned}[t,X(w)] &= [s\cdot\sigma,-\kappa_n(\sigma^{-1})\kappa_n(\mathrm{d}\sigma X\cdot\sigma^{-1})(v)+\kappa_n(\sigma^{-1})(X(v))]\\ &= [s,X(v)]-[s,\kappa_n(\mathrm{d}\sigma X\cdot\sigma^{-1})(v)]\,.\end{aligned} \tag{3.3}$$

In order to obtain an expression for $\mathrm{d}\sigma X \cdot \sigma^{-1}$ we will first calculate $\lambda_*(\mathrm{d}\sigma X \cdot \sigma^{-1})$. Denote $A = (A_{ij}) = \lambda \circ \sigma : W \to \mathrm{SO}(n)$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve with $\gamma(0) = x \in W$ and $\gamma'(0) = X \in T_x M$. Then

$$\begin{aligned}
\lambda_*(\mathrm{d}\sigma X \cdot \sigma^{-1}) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \lambda(\sigma \circ \gamma(t) \cdot \sigma(x)^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} (\lambda \circ \sigma \circ \gamma)(t) \cdot \lambda(\sigma(x)^{-1}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} A \circ \gamma(t) \cdot A^t = \mathrm{d}AX \cdot A^t \\
&= \sum_{k=1}^{n} (X(A_{ik}) A_{jk}) \\
&= \frac{1}{2} \sum_{i,j,k=1}^{n} X(A_{ik}) A_{jk} X_{e_i, e_j} ,
\end{aligned}$$

where the X_{e_i,e_j} are the matrices from Exercise 5. By Proposition 1.47 we now have

$$\mathrm{d}\sigma X \cdot \sigma^{-1} = \frac{1}{4} \sum_{i,j,k=1}^{n} X(A_{ik}) A_{jk} e_i \cdot e_j . \tag{3.4}$$

Next, we consider the second term on the right-hand side of (3.2). Recall that the tangent bundle is (isomorphic to) the vector bundle $P \times_\lambda \mathbb{R}^n$ associated with the principal fibre bundle P of our Spin-structure and the representation λ. With $(\mathbf{e}_1, \dots, \mathbf{e}_n) = \pi \circ s$ and $(\mathbf{f}_1, \dots, \mathbf{f}_n) = \pi \circ t$ the projected local OONBs, we have for each $i = 1, \dots, n$,

$$\begin{aligned}
\mathbf{f}_i &= [t, e_i] = [s \cdot \sigma, e_i] = [s, \lambda(\sigma) e_i] = [s, A e_i] = \left[s, \sum_{j=1}^{n} A_{ji} e_j \right] = \sum_{k=1}^{n} A_{ji} [s, e_j] \\
&= \sum_{k=1}^{n} A_{ji} \mathbf{e}_j ,
\end{aligned}$$

which implies

$$\nabla^{\mathrm{LC}}_X \mathbf{f}_i = \sum_{j=1}^{n} \nabla^{\mathrm{LC}}_X (A_{ji} \mathbf{e}_j) = \sum_{j=1}^{n} X(A_{ji}) \mathbf{e}_j + \sum_{j=1}^{n} A_{ji} \nabla^{\mathrm{LC}}_X \mathbf{e}_j .$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi &= \sum_{i,j,k=1} A_{ji} \mathbf{e}_j \cdot \Big(X(A_{ki}) \mathbf{e}_k + A_{ki} \nabla^{\mathrm{LC}}_X \mathbf{e}_k \Big) \cdot \varphi \\
&= \sum_{i,j,k=1} X(A_{ki}) A_{ji} \mathbf{e}_j \cdot \mathbf{e}_k \cdot \varphi + \sum_{i,j,k=1} A_{ji} A_{ki} \mathbf{e}_j \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_k \cdot \varphi \\
&= \left[s, \kappa_n \left(\sum_{i,j,k=1} X(A_{ki}) A_{ji} e_j \cdot e_k \right) (v) \right] + \sum_{i,j,k=1} A_{ji} A_{ki} \mathbf{e}_j \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_k \cdot \varphi .
\end{aligned}$$

Since $A^{-1} = A^t$ we have $\sum_i A_{ji} A_{ki} = \delta_{kl}$ and using (3.4) we obtain

$$\sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi = 4[s, \kappa_n(\mathrm{d}\sigma X \cdot \sigma^{-1})] + \sum_{i=1} \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi ,$$

which in turn, using (3.3), implies

$$[s, X(v)] + \frac{1}{4} \sum_{i=1}^{n} \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi = [t, X(w)] + \frac{1}{4} \sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi .$$

Step 3: We have to show that our connection is metric and satisfies (3.1). To see that ∇ is metric, let $s : U \to P$ be a local section with $(\mathbf{e}_1, \dots, \mathbf{e}_n) = \pi \circ s : U \to \mathrm{SO}(M, g)$ the accompanying OONB, $\varphi = [s, v], \psi = [s, w] \in$

$\Gamma(U, \Sigma M)$ with $v, w \in C^\infty(U, \Sigma_n)$ and $X \in T_x M$. Then, by definition of the bundle metric, see Proposition 2.23, we have

$$X\langle \varphi, \psi \rangle = X\langle v, w \rangle = \langle X(v), w \rangle + \langle v, X(w) \rangle = \langle [s, X(v)], \psi \rangle + \langle \varphi, [s, X(w)] \rangle .$$

Using the skew-symmetry of Clifford multiplication, that the Levi-Civita connection is metric and the Clifford relations, we see that

$$\begin{aligned} \langle \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi, \psi \rangle + \langle \varphi, \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \psi \rangle &= \langle \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \mathbf{e}_i \cdot \varphi, \psi \rangle \\ &= -2g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) \langle \varphi, \psi \rangle , \end{aligned}$$

which vanishes since

$$0 = Xg(\mathbf{e}_i, \mathbf{e}_i) = 2g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) .$$

Hence,

$$X\langle \varphi, \psi \rangle = \langle [s, X(v)], \psi \rangle + \langle \varphi, [s, X(w)] \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle .$$

To see that ∇ satisies (3.1) we let $Y = [s, y] \in \Gamma(U, TM)$ with $y \in C^\infty(U, \mathbb{R}^n)$. Observe that

$$Y = [s, y] = \left[s, \sum_{i=1}^n y_i e_i \right] = \sum_{i=1}^n y_i [s, e_i] = \sum_{i=1}^n g(Y, \mathbf{e}_i) \mathbf{e}_i$$

and

$$Y \cdot \varphi = [s, y] \cdot [s, v] = [s, \kappa_n(y)(v)] .$$

Thus

$$\nabla_X (Y \cdot \varphi) = [s, X(\kappa_n(y)(v))] + \frac{1}{4} \sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi . \tag{3.5}$$

The first term on the right-hand side is

$$\begin{aligned} X(\kappa_n(y)(v)) &= X(\kappa_n(y))(v) + \kappa_n(y)(X(v)) = \kappa_n(X(y))(v) + \kappa_n(y)(X(v)) \\ &= \sum_{i=1}^n X(y_i) \kappa_n(e_i)(v) + \kappa_n(y)(X(v)) = \sum_{i=1}^n X(g(\mathbf{e}_i, Y)) \kappa_n(e_i)(v) + \kappa_n(y)(X(v)) , \end{aligned}$$

so that

$$[s, X(\kappa_n(y)(v))] = \sum_{i=1}^n X(g(\mathbf{e}_i, Y)) \mathbf{e}_i \cdot \varphi + Y \cdot [s, X(v)] . \tag{3.6}$$

Using the Clifford relations, we see that the second term on the right-hand side of (3.5) is

$$\begin{aligned} \sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi &= -\sum_{i=1}^n \mathbf{e}_i \cdot Y \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi - 2 \sum_{i=1}^n g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, Y) \mathbf{e}_i, Y) \cdot \varphi \\ &= \sum_{i=1}^n Y \cdot \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + 2 \sum_{i=1}^n g(\mathbf{e}_i, Y) \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi - 2 \sum_{i=1}^n g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, Y) \mathbf{e}_i \cdot \varphi . \end{aligned}$$

Since the Levi-Civita connection is metric, for each $j = 1, \ldots, n$ we have

$$\begin{aligned} \sum_{i=1}^n g(g(\mathbf{e}_i, Y) \nabla^{\mathrm{LC}}_X \mathbf{e}_i, \mathbf{e}_j) &= \sum_{i=1}^n g(\mathbf{e}_i, Y) g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, \mathbf{e}_j) = -\sum_{i=1}^n g(\mathbf{e}_i, Y) g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_j) = -g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_j) \\ &= -\sum_{i=1}^n g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) g(\mathbf{e}_i, \mathbf{e}_j) = -\sum_{i=1}^n g(g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) \mathbf{e}_i, \mathbf{e}_j) , \end{aligned}$$

which implies

$$\frac{1}{4} \sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi = \frac{1}{4} Y \cdot \sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + \sum_{i=1}^n g(\mathbf{e}_i, Y) \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi .$$

From this, (3.5) and (3.6) we obtain

$$\begin{aligned}\nabla_X(Y\cdot\varphi) &= \sum_{i=1}^n X(g(\mathbf{e}_i,Y))\mathbf{e}_i\cdot\varphi + \sum_{i=1}^n g(\mathbf{e}_i,Y)\nabla_X^{\mathrm{LC}}\mathbf{e}_i\cdot\varphi + Y\cdot\nabla_X\varphi\\ &= \nabla_X^{\mathrm{LC}}\left(\sum_{i=1}^n g(\mathbf{e}_i,Y)\mathbf{e}_i\right)\cdot\varphi + Y\cdot\nabla_X\varphi\\ &= \nabla_X^{\mathrm{LC}}Y\cdot\varphi + Y\cdot\nabla_X\varphi\,.\end{aligned}$$

□

Remark 3.14. *On any Riemannian manifold (M,g) there are vector bundle isomorphisms*

$$TM \underset{\sharp}{\overset{\flat}{\rightleftarrows}} T^*M$$

called musical isomorphisms which are given by the metric, i.e., for any $x\in M$ and $X\in T_xM$ we have

$$T_xM \ni X \mapsto X^\flat \in T_x^*M$$

with

$$X^\flat(Y) := g_x(X,Y)$$

and

$$\sharp = \flat^{-1}\,.$$

Definition 3.15. *Let (M,g) be a Riemannian spin manifold with Spin-structure (P,π), associated spinor bundle ΣM and Clifford multiplication $\mu : TM\otimes\Sigma M\to\Sigma M$. The Dirac operator D is the 1st order linear differential operator*

$$\boxed{D:\Gamma(M,\Sigma M)\xrightarrow{\nabla}\Gamma(M,T^*M\otimes\Sigma M)\xrightarrow{\sharp\otimes\mathrm{id}}\Gamma(M,TM\otimes\Sigma M)\xrightarrow{\mu}\Gamma(M,\Sigma M)\,.}$$

Proposition 3.16. *Let $(e_1,\dots,e_n)$ be a local ONB. Then the Dirac operator is given by*

$$\boxed{D\varphi = \sum_{i=1}^n e_i\cdot\nabla_{e_i}\varphi}$$

for all $\varphi\in\Gamma(M,\Sigma M)$. Moreover, we have

$$D(f\varphi) = \operatorname{grad} f\cdot\varphi + fD\varphi$$

for all $f\in C^\infty(M,\mathbb{C})$ and $\varphi\in\Gamma(M,\Sigma M)$, where $\operatorname{grad} f := (\mathrm{d}f)^\sharp$.

Proof. Let $\varepsilon_i = e_i^\flat$ for all $i=1,\dots,n$. Then

$$\nabla\varphi = \sum_{i=1}^n \varepsilon_i\otimes\nabla_{e_i}\varphi\,,$$

so that

$$D\varphi = \mu\circ(\sharp\otimes\mathrm{id})\left(\sum_{i=1}^n\varepsilon_i\otimes\nabla_{e_i}\varphi\right) = \mu\left(\sum_{i=1}^n e_i\otimes\nabla_{e_i}\varphi\right) = \sum_{i=1}^n e_i\cdot\nabla_{e_i}\varphi\,.$$

Using the formula we just proved, we see that

$$D(f\varphi) = \sum_{i=1}^n e_i\cdot\nabla_{e_i}(f\varphi) = \sum_{i=1}^n e_i\cdot(e_i(f)\varphi + f\nabla_{e_i}\varphi) = \sum_{i=1}^n e_i(f)e_i\cdot\varphi + \sum_{i=1}^n e_i\cdot\nabla_{e_i}\varphi = \operatorname{grad} f\cdot\varphi + fD\varphi\,.$$

□

Definition 3.17. *Let (M,g) be a Riemannian manifold.*

(i) *Denote by $\mathcal{B}(M)$ the Borel σ-algebra of M, i.e., the smallest σ-algebra containing all open sets of M. We define the Riemannian measure / volume $\mu := \mu_g$ on M to be the measure which in every chart (U, x) is given by*

$$\mathrm{d}\mu := \sqrt{\det(g_{ij})}\mathrm{d}\lambda \,,$$

where λ is the Lebesgue-measure in (U, x) and

$$g_{ij} := g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right) \quad \text{for} \quad i, j = 1, \ldots, n \,,$$

are the components of the matrix of g associated with the coordinates $(x^1, \ldots, x^n)$.

(ii) *Let $(E, \pi_E; V)$ be any $\mathbb{K}$-vector bundle over M and $\varphi \in \Gamma(M, E)$. The support of φ is the set*

$$\operatorname{supp}\varphi := \overline{\{x \in M \,|\, \varphi(x) \neq 0\}} \,.$$

We say that φ is compactly supported if $\operatorname{supp}\varphi$ is compact and denote the space of all compactly supported sections by

$$\Gamma_c(M; E) := \{\varphi \in \Gamma(M, E) \,|\, \operatorname{supp}\varphi \text{ is compact}\} \,.$$

In the case of $E = TM$ we additionally introduce the notation

$$\mathcal{V}_c(M) := \Gamma_c(M, TM) \,.$$

(iii) *Suppose that $(E, \pi_E; V)$ comes equipped with a bundle metric $\langle \cdot, \cdot \rangle$. We define the L^2-inner product $(\cdot, \cdot) := (\cdot, \cdot)_{L^2}$ on $\Gamma_c(M; E)$ by*

$$(\varphi, \psi)_{L^2} := \int_M \langle \varphi, \psi \rangle \mathrm{d}\mu_g$$

and the associated L^2-norm $|\cdot| := |\cdot|_{L^2}$ by

$$|\varphi|_{L^2} := \sqrt{(\varphi, \varphi)} \,.$$

Remark 3.18. *Note that $\Gamma_c(M; E)$ is in general not* complete *w.r.t. $|\cdot|_{L^2}$, i.e., the pair $(\Gamma_c(M; E), (\cdot, \cdot))$ is only a pre-Hilbert space.*

Definition 3.19. *Let (M, g) be a Riemannian manifold and $X \in \mathcal{V}(M)$ a vector field. The divergence of X is the function $\operatorname{div} X \in C^\infty(M)$ given locally by*

$$\operatorname{div} X = \sum_{i=1}^n g(e_i, \nabla_{e_i} X) = \operatorname{tr}_g(\nabla X) \,,$$

where $(e_1, \ldots, e_n)$ is a local ONB.

The familiar Divergence Theorem from vector calculus generalizes to Riemannian manifolds and we state it here without proof.

Theorem 3.20. *Let (M, g) be a Riemannian manifold and $X \in \mathcal{V}_c(M)$. Then*

$$\int_M \operatorname{div} X \mathrm{d}\mu_g = 0 \,.$$

Notation and Remarks 3.21. *We denote by $TM^{\mathbb{C}}$ the complexification of the tangent bundle. Formally, this is the complex vector bundle over M given by*

$$TM^{\mathbb{C}} = \bigcup_{x \in M} (T_x M)^{\mathbb{C}}$$

where $(T_x M)^{\mathbb{C}} = T_x M \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $T_x M$. Each element $z \in (T_x M)^{\mathbb{C}}$ can be written as

$$z = v + iw \quad \text{with} \quad v, w \in T_x M \,.$$

We denote $\mathcal{V}^{\mathbb{C}}_{(c)}(M) := \Gamma_{(c)}(M, TM^{\mathbb{C}})$ and call its elements complex (compactly supported) vector fields. Each element $Z \in \mathcal{V}^{\mathbb{C}}(M)$ can be written in the form

$$Z = V + iW \quad \text{for suitable} \quad V, W \in \mathcal{V}(M) \,.$$

We extend the Levi-Civita connection ∇ complex linearly to a connection of $TM^{\mathbb{C}}$, denoted by the same symbol, and we do the same with the divergence. The Divergence Theorem is then of course also true for all complex compactly supported vector fields.

Proposition 3.22. *Let (M,g) be an oriented Riemannian spin manifold with a fixed Spin-structure. Then the Dirac operator is formally selfadjoint, i.e., we have*

$$(D\varphi,\psi)=(\varphi,D\psi)\quad \textit{for all}\quad \varphi,\psi\in\Gamma_c(M;\Sigma M)\,.$$

Proof. Let $p\in M$ and $(e_1,\dots,e_n)$ be an ONB defined in a neighborhood of p with $(\nabla e_i)_p=0$. Then at p we have

$$\begin{aligned}\langle D\varphi,\psi\rangle_p &= \sum_{i=1}^n\langle e_i\cdot\nabla_{e_i}\varphi,\psi\rangle_p = -\sum_{i=1}^n\langle\nabla_{e_i}\varphi,e_i\cdot\psi\rangle\\ &= -\sum_{i=1}^n\big((e_i)_p\langle\varphi,e_i\cdot\psi\rangle-\langle\varphi,\nabla_{e_i}e_i\cdot\psi\rangle_p-\langle\varphi,e_i\cdot\nabla_{e_i}\psi\rangle_p\big)\\ &= -\sum_{i=1}^n\big((e_i)_p\langle\varphi,e_i\cdot\psi\rangle-\langle\varphi,e_i\cdot\nabla_{e_i}\psi\rangle_p\big)\\ &= -\sum_{i=1}^n(e_i)_p\langle\varphi,e_i\cdot\psi\rangle+\langle\varphi,D\psi\rangle_p\,.\end{aligned}$$

Define a complex compactly supported vector field $X\in\mathcal{V}_c^{\mathbb{C}}(M)$ by the condition

$$(g_x\otimes\mathrm{id})(X_x,W)=-\langle\varphi(x),W\cdot\psi(x)\rangle_x\quad\text{for all}\quad W\in T_xM,x\in M\,.$$

Then

$$\begin{aligned}\operatorname{div}X(p) &= \sum_{i=1}^n(g\otimes\mathrm{id})(\nabla_{e_i}X,e_i)_p=\sum_{i=1}^n\big((e_i)_p(g\otimes\mathrm{id})(X,e_i)-(g\otimes\mathrm{id})(X,\nabla_{e_i}e_i)_p\big)\\ &= \sum_{i=1}^n(e_i)_p(g\otimes\mathrm{id})(X,e_i)=-\sum_{i=1}^n(e_i)_p\langle\varphi,e_i\cdot\psi\rangle,\end{aligned}$$

from which we deduce

$$\langle D\varphi,\psi\rangle=\operatorname{div}X+\langle\varphi,D\psi\rangle\,.$$

The statement of the theorem now follows from the Divergence Theorem. □

Corollary 3.23. *Let (M,g) be a compact Riemannian spin manifold with a fixed Spin-structure. Then*

$$\ker D=\ker D^2\,.$$

Remark 3.24. *We call any spinor $\varphi\in\Gamma(M,\Sigma M)$ with $D^2\varphi=0$ harmonic and in case M is compact, this is equivalent to $D\varphi=0$.*

proof of Corollary 3.23. We only need to show $\ker D^2\subseteq\ker D$. Let $\varphi\in\ker D^2$, i.e., $D^2\varphi=0$. Then we also have $(D^2\varphi,\varphi)=0$. Hence,

$$0=(D^2\varphi,\varphi)=(D\varphi,D\varphi)=\int_M\langle D\varphi,D\varphi\rangle\mathrm{d}\mu_g\,.$$

The integrand is a nonnegative, continuous function. We claim that it must be zero. Assume it is not, i.e., there is a point $p\in M$ such that $\langle D\varphi,D\varphi\rangle_p>0$. By continuity, there is an open neighborhood of p on which this function is positive. Since the Riemannian measure is of full support (every open set has positive measure), the integral would be positive. A contradiction. Hence, $\langle D\varphi,D\varphi\rangle\equiv0$ which implies $D\varphi=0$. □

3.1. **The Lichnerowicz formula.** The goal of this section is to come back to the very first lecture and see that, in a suitable sense, the square of the Dirac operator is a Laplacian. The corresponding formula is called the Lichnenrowicz formula (see Theorem 3.31) and it shows that there is an interesting interplay between the geometry of a manifold and the existence of harmonic spinors, i.e., solutions to the equation $D^2\varphi=0$.

Let (M,g) be a Riemannian manifold. Recall the definition of the Riemannian curvature tensor

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z\,,$$

the Ricci curvature tensor

$$\operatorname{Ric}(X,Y)=\sum_{i=1}^n g(R(e_i,X)Y,e_i)=\operatorname{tr}(U\mapsto R(U,X)Y)\,,$$

and the scalar curvature

$$\mathrm{scal} = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i) = \mathrm{tr}_g((U,V) \mapsto \mathrm{Ric}(U,V)) = \sum_{i,j=1}^{n} g(R(e_i,e_j)e_j,e_i)\,.$$

The Riemannian curvature tensor has the following symmetry properties,

$$\begin{aligned} R(X,Y)Z &= -R(Y,X)Z\,,\\ g(R(X,Y)Z,W) &= -g(R(X,Y)W,Z)\,,\\ g(R(X,Y)Z,W) &= g(R(Z,W)X,Y)\,,\\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0\,. \end{aligned}$$

The last equation is 1st Bianchi-identity.

It follows from the symmetry properties of the Riemannian curvature tensor, that the Ricci tensor is symmetric, i.e., $\mathrm{Ric}(X,Y) = -\mathrm{Ric}(Y,X)$. It thus defines, by duality, a (pointwise) selfadjoint endomorphism field ric,

$$g(\mathrm{ric}(X),Y) = \mathrm{Ric}(X,Y)\,.$$

Definition 3.25. *Let M be a manifold and $(E,\pi_E;V)$ a $\mathbb{K}$-vector bundle over M equipped with a connection $\nabla^E : \Gamma(M,E) \to \Gamma(M,T^*M \otimes E)$. We define the curvature tensor R^E of (E,∇^E) by*

$$R^E(X,Y)\varphi = \nabla^E_X\nabla^E_Y\varphi - \nabla^E_Y\nabla^E_X\varphi - \nabla^E_{[X,Y]}\varphi \quad \textit{for all} \quad X,Y \in \mathcal{V}(M), \varphi \in \Gamma(M,E)\,.$$

Remark 3.26. *A calculation completely analogous to the one for the Riemannian curvature tensor shows that R^E is indeed C^∞-linear in all three arguments so that it is indeed a tensor, i.e., a section $R^E \in \Gamma(M, T^*M \otimes T^*M \otimes \mathsf{End}(E))$, and that it is antisymmetric in the first two arguments, i.e., $R^E(X,Y)\sigma = -R^E(Y,X)\sigma$ for all $X,Y \in T_xM$, $\sigma \in E_x$, $x \in M$.*

Proposition 3.27. *Let (M,g) be a Riemannian spin manifold with a fixed Spin-structure (P,π). Then*

$$R^{\Sigma M}(X,Y)\sigma = \frac{1}{4}\sum_{i=1}^{n} e_i \cdot R(X,Y)e_i \cdot \sigma\,,$$

where $(e_1,\ldots,e_n)$ is an ONB of the corresponding tangent space.

Proof. Let $p \in M$ and let $(e_1,\ldots,e_n)$ be a local OONB defined on a neighborhood U of p with $(\nabla e_i)_p = 0$ for all $i = 1,\ldots,n$. Choose a section $s : U \to P$ such that $\pi \circ s = (e_1,\ldots,e_n)$. Let $X,Y \in \mathcal{V}(M)$, $v \in C^\infty(U,\Sigma_n)$ and let $\varphi = [s,v] \in \Gamma(U;\Sigma M)$. Then we have (cmp. the proof of Theorem 3.13, Step 1)

$$\begin{aligned} \nabla^\Sigma_X\nabla^\Sigma_Y\varphi &= \nabla^\Sigma_X\left([s,Y(v)] + \tfrac14\sum_{i=1} e_i\cdot\nabla_Y e_i\cdot\varphi\right)\\ &= [s,X(Y(v))] + \sum_{i=1}^{n} e_i\cdot\nabla_X e_i\cdot[s,Y(v)] + \tfrac14\sum_{i=1}^{n}\nabla^\Sigma_X(e_i\cdot\nabla_Y e_i\cdot\varphi)\\ &= [s,X(Y(v))] + \sum_{i=1}^{n} e_i\cdot\nabla_X e_i\cdot[s,Y(v)] + \tfrac14\sum_{i=1}^{n}\left(\nabla_X e_i\cdot\nabla_Y e_i\cdot\varphi + e_i\cdot\nabla_X\nabla_Y e_i\cdot\varphi + e_i\cdot\nabla_Y e_i\cdot\nabla^\Sigma_X\varphi\right)\,. \end{aligned}$$

Analogously, we have

$$\nabla^\Sigma_Y\nabla^\Sigma_X\varphi = [s,Y(X(v))] + \sum_{i=1}^{n} e_i\cdot\nabla_Y e_i\cdot[s,X(v)] + \tfrac14\sum_{i=1}^{n}\left(\nabla_Y e_i\cdot\nabla_X e_i\cdot\varphi + e_i\cdot\nabla_Y\nabla_X e_i\cdot\varphi + e_i\cdot\nabla_X e_i\cdot\nabla^\Sigma_Y\varphi\right)\,,$$

and also

$$\nabla^\Sigma_{[X,Y]}\varphi = [s,[X,Y](v)] + \tfrac14\sum_{i=1}^{n} e_i\cdot\nabla_{[X,Y]}e_i\cdot\varphi\,,$$

so that, at the point p, we have

$$R^{\Sigma M}(X_p,Y_p)(\varphi(p)) = \tfrac14\sum_{i=1}^{n}(e_i)_p R(X_p,Y_p)(e_i)_p\cdot\varphi(p)\,,$$

as claimed. $\square$

Definition 3.28. *Let (M,g) be a Riemannian manifold and $(E,\pi_E;V)$ a $\mathbb{K}$-vector bundle over M, equipped with a connection ∇^E. The associated Bochner Laplacian , also called the connection Laplacian , is the linear second order differential operator*

$$\Delta^E : \Gamma(M,E) \to \Gamma(E;M)$$
$$\varphi \mapsto -\sum_{i=1}^n \left(\nabla^E_{e_i}\nabla^E_{e_i}\varphi - \nabla^E_{\nabla_{e_i}e_i}\varphi\right),$$

where $(e_1,\dots,e_n)$ is a local ONB. In case M is a spin manifold and $E = \Sigma M$ is the spinor bundle associated with a Spin-structure, we call $\Delta^\Sigma := \Delta^{\Sigma M}$ the spinor Laplacian .

Proposition 3.29. *Let (M,g) be a Riemannian manifold and $(E,\pi_E;V)$ a $\mathbb{K}$-vector bundle with a bundle metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇^E. Then the associated Bochner Laplacian satisfies*

$$(\Delta^E\varphi,\psi) = (\nabla^E\varphi,\nabla^E\psi) \quad \textit{for all} \quad \varphi,\psi \in \Gamma_c(M;E)\,.$$

In particular, Δ^E is nonnegative and formally self-adjoint, i.e.,

$$(\Delta^E\varphi,\varphi) \geqslant 0 \quad \textit{and} \quad (\Delta^E\varphi,\psi) = (\varphi,\Delta^E\psi) \quad \textit{for all} \quad \varphi,\psi \in \Gamma_c(M;E)\,.$$

Remark. *The expression $|\nabla\varphi|^2$ has to be read as follows. The Riemannian metric g induces a bundle metric g^* on T^*M by*

$$g^*_x(\alpha,\beta) = g_x(\alpha^\sharp,\beta^\sharp) \quad \textit{for all} \quad \alpha,\beta \in T^*_xM, x \in M\,.$$

The bundle metric g^* is sometimes called the cometric . Now we can use the tensor product metric $\langle\cdot,\cdot\rangle_\otimes$ on $T^*M\otimes E$ which is given on pure tensors by

$$\langle\alpha\otimes\sigma,\beta\otimes\tau\rangle_{\otimes x} := g^*_x(\alpha,\beta)\langle\sigma,\tau\rangle_x \quad \textit{for all} \quad \alpha,\beta \in T^*_xM, \sigma,\tau \in E_x, x \in M\,.$$

Then $|\nabla\varphi|^2$ is the square of the corresponding L^2-norm of $\nabla\varphi$.

Proof. As before, we fix a point $p \in M$ and choose a local ONB $(e_1,\dots,e_n)$ defined on a neighborhood of p with $(\nabla e_i)_p = 0$ for all $i = 1,\dots,n$. Then at p we have

$$\begin{aligned}
\langle\Delta^E\varphi,\psi\rangle_p &= -\sum_{i=1}^n\langle\nabla_{e_i}\nabla_{e_i}\varphi,\psi\rangle_p = -\sum_{i=1}^n\left(e_i\langle\nabla^E_{e_i}\varphi,\psi\rangle - \langle\nabla^E_{e_i}\varphi,\nabla^E_{e_i}\psi\rangle\right)_p \\
&= -\sum_{i=1}^n(e_i)_p\langle\nabla^E_{e_i}\varphi,\psi\rangle + \sum_{i,j=1}^n g(e_i,e_j)_p\langle\nabla^E_{e_i}\varphi,\nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n(e_i)_p\langle\nabla^E_{e_i}\varphi,\psi\rangle + \sum_{i,j=1}^n g^*(\varepsilon_i,\varepsilon_j)_p\langle\nabla^E_{e_i}\varphi,\nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n(e_i)_p\langle\nabla^E_{e_i}\varphi,\psi\rangle + \sum_{i,j=1}^n \langle\varepsilon_i\otimes\nabla^E_{e_i}\varphi,\varepsilon_j\otimes\nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n(e_i)_p\langle\nabla^E_{e_i}\varphi,\psi\rangle + \langle\nabla^E\varphi,\nabla^E\psi\rangle_{\otimes p}\,.
\end{aligned}$$

In case E is a real vector bundle, we define a compactly supported vector field $X \in \mathcal{V}_c(M)$ by

$$g_x(X_x,W) = -\langle\nabla^E_W\varphi(x),\psi(x)\rangle_x \quad \text{for all} \quad W \in T_xM, x \in M\,,$$

and in case E is complex we substitute $g\otimes\mathrm{id}$ for g to define X as a complex compactly supported vector field. In both cases, a calculation analogous to the one in the proof of Proposition 3.22 shows that

$$\operatorname{div} X(p) = -\sum_{i=1}^n(e_i)_p\langle\nabla^E_{e_i}\varphi,\psi\rangle\,.$$

Hence, it follows from the Divergence Theorem that

$$(\Delta^E\varphi,\psi) = (\nabla^E\varphi,\nabla^E\psi)\,.$$

Nonnegativity now follows by setting $\psi = \varphi$ and formal selfadjointness of Δ^E follows straightforwardly,

$$(\Delta^E\varphi,\psi) = (\nabla^E\varphi,\nabla^E\psi) = \overline{(\nabla^E\psi,\nabla^E\varphi)} = \overline{(\Delta^E\psi,\varphi)} = (\varphi,\Delta^E\psi)\,.$$

□

Corollary 3.30. *In the situation of Proposition 3.29, every $\varphi \in \Gamma_c(M;E)$ which is Δ^E-harmonic , i.e., satisfies $\Delta^E\varphi = 0$, is parallel , i.e., satisfies $\nabla^E\varphi \equiv 0$.*

Proof. Let $\varphi \in \Gamma_c(M;E)$ be harmonic. Since $\Delta^E\varphi = 0$, we also have $(\Delta^E\varphi,\varphi) = 0$. By the last proposition,

$$0 = (\Delta^E\varphi,\varphi) = (\nabla^E\varphi,\nabla^E\varphi) = \int_M \langle\nabla^E\varphi,\nabla^E\varphi\rangle \mathrm{d}\mu_g\,.$$

The same argument as in the proof of Corollary 3.23 shows that $\nabla^E\varphi = 0$. □

Theorem 3.31 (Lichnerowicz formula)**.** *Let (M,g) be a Riemannian spin manifold with a fixed Spin-structure. Then we have*

$$\boxed{D^2\varphi = \Delta^\Sigma\varphi + \frac{1}{4}\operatorname{scal}\cdot\varphi \quad \textit{for all} \quad \varphi\in\Gamma(M,\Sigma M)\,.}$$

Proof. Let $p \in M$ and choose a local ONB $(e_1,\dots,e_n)$ with $(\nabla e_i)_p = 0$ for all $i = 1,\dots,n$. Then, at p, we have

$$\begin{aligned} D^2\varphi &= \sum_{i,j=1}^n e_i\cdot\nabla_{e_i}\left(e_j\cdot\nabla_{e_j}\varphi\right) = \sum_{i,j=1}^n e_i\cdot\left(\nabla_{e_i}e_j\cdot\nabla_{e_j}\varphi + e_j\cdot\nabla_{e_i}\nabla_{e_j}\varphi\right) = \sum_{i,j=1} e_i\cdot e_j\cdot\nabla_{e_i}\nabla_{e_j}\varphi \\ &= -\sum_{i=1}^n\nabla_{e_i}\nabla_{e_i}\varphi + \sum_{i<j} e_i\cdot e_j\cdot\left(\nabla_{e_i}\nabla_{e_j}\varphi - \nabla_{e_j}\nabla_{e_i}\varphi\right)\,. \end{aligned}$$

Since $(\nabla e_i)_p = 0$ and $[e_i,e_j]_p = (\nabla_{e_i}e_j - \nabla_{e_j}e_i)_p = 0$ (the Levi-Civita connection is, by definition, torsionfree), this is equal to

$$\begin{aligned} &-\sum_{i=1}^n\left(\nabla_{e_i}\nabla_{e_i}\varphi - \nabla_{\nabla_{e_i}e_i}\varphi\right) + \sum_{i<j}e_i\cdot e_j\cdot\left(\nabla_{e_i}\nabla_{e_j}\varphi - \nabla_{e_j}\nabla_{e_i}\varphi - \nabla_{[e_i,e_j]}\varphi\right) \\ =&\Delta^\Sigma\varphi + \sum_{i<j}e_i\cdot e_j\cdot R^{\Sigma M}(e_i,e_j)\varphi = \Delta^\Sigma\varphi + \frac{1}{2}\sum_{i,j=1}^n e_i\cdot e_j\cdot R^{\Sigma M}(e_i,e_j)\varphi\,. \end{aligned}$$

It remains to show that the second term on the right hand side is equal to $1/4\operatorname{scal}\varphi$. By Proposition 3.27 this term is

$$\begin{aligned} &\frac{1}{8}\sum_{i,j,k=1}^n e_i\cdot e_j\cdot e_k\cdot R(e_i,e_j)e_k\cdot\varphi = \frac{1}{8}\sum_{i,j,k,l=1} g(R(e_i,e_j)e_k,e_l)e_i\cdot e_j\cdot e_k\cdot e_l\cdot\varphi \\ &= \frac{1}{8}\sum_{l=1}^n\Bigg(\frac{1}{3}\sum_{\substack{i,j,k\\ \text{p.w. dist.}}} g(R(e_i,e_j)e_k + R(e_j,e_k)e_i + R(e_k,e_i)e_j,e_l)e_i\cdot e_j\cdot e_k\cdot \\ &+\sum_{i,j=1}^n g(R(e_i,e_j)e_i,e_l)e_i\cdot e_j\cdot e_i\cdot + \sum_{i,j=1}^n g(R(e_i,e_j)e_j,e_l)e_i\cdot e_j\cdot e_j\cdot\Bigg)e_l\cdot\varphi\,. \end{aligned}$$

By the first Bianchi-identity for the Riemannian curvature tensor, the first sum vanishes and we are left with

$$\begin{aligned} &\frac{1}{8}\sum_{l=1}^n\left(\sum_{i,j=1}^n g(R(e_i,e_j)e_l,e_i)e_j\cdot e_i\cdot e_i\cdot + \sum_{i,j=1}^n g(R(e_j,e_i)e_l,e_j)e_i\cdot e_j\cdot e_j\cdot\right)e_l\cdot\varphi \\ &= -\frac{1}{4}\sum_{i,l=1}^n\operatorname{Ric}(e_i,e_l)e_i\cdot e_l\cdot\varphi = -\frac{1}{4}\sum_{i=1}^n\operatorname{Ric}(e_i,e_i)e_i\cdot e_i\cdot\varphi = \frac{1}{4}\operatorname{scal}\varphi\,, \end{aligned}$$

where we have used the symmetry properties of the curvature tensor, the Ricci curvature and the Clifford relations. □

Corollary 3.32. *Let (M, g) be a connected, compact Riemannian spin manifold with fixed Spin-structure. Assume that* $\mathrm{scal} \geqslant 0$ *and that there exists a point $p \in M$ such that* $\mathrm{scal}(p) > 0$. *Then there do not exist any nontrivial harmonic spinors, i.e., the equation*

$$D\varphi = 0\,, \qquad \varphi \in \Gamma(M, \Sigma M)$$

has only the trivial solution.

Proof. Let $\varphi \in \Gamma(M, \Sigma M)$ be a harmonic spinor. Then $D^2\varphi = 0$ and so

$$0 = (D^2\varphi, \varphi) = (\Delta^\Sigma \varphi, \varphi) + \tfrac{1}{4}(\mathrm{scal}\,\varphi, \varphi)\,,$$

that is,

$$-|\nabla\varphi|^2 = -(\nabla\varphi, \nabla\varphi) = -(\Delta^\Sigma\varphi, \varphi) = \tfrac{1}{4}(\mathrm{scal}\,\varphi, \varphi) = \tfrac{1}{4}\int_M \mathrm{scal}\,\|\varphi\|^2 \mathrm{d}\mu_g\,.$$

The right-hand side is nonnegative, so we must have $\nabla\varphi = 0$. Since the spinor connection is metric, this implies that $\|\varphi\|^2$ is constant,

$$X\|\varphi\|^2 = X\langle\varphi, \varphi\rangle = \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle = 0 + 0 \quad \text{for all} \quad X \in T_xM, x \in M\,.$$

By assumption $\mathrm{scal}(p) > 0$ which means we must have $\mathrm{scal} > 0$ on an open neighborhood of p. This implies $\|\varphi\|^2 = 0$ for otherweise the integral on the right hand-side was positive. □

3.2. **Special Spinors and Geometry.** We constructed the spinor bundle and its covariant derivative using the metric and the Levi-Civita connection. This means that the geometry of the spinor bundle is closely related to the geometry of the underlying manifold, a fact which can be seen in the formula for the curvature tensor of ΣM or in the Lichnerowicz-formula. It comes as no surprise that the existence of spinors satisfying certain field equations has strong geometric implications.

Definition 3.33. *Let (M, g) be a Riemannian spin manifold with a fixed Spin-structure. Then a spinor $\varphi \in \Gamma(M, \Sigma M)$ is called parallel if*

$$\nabla\varphi = 0\,,$$

that is, if $\nabla_X\varphi = 0$ for all $X \in \mathcal{V}(M)$.

Lemma 3.34. *If M is connected and $\varphi \in \Gamma(M, \Sigma M)$ parallel, then the function $\|\varphi\|$ is constant.*

Proof. We have for every $X \in \mathcal{V}(M)$,

$$X\|\varphi\|^2 = X\langle\varphi, \varphi\rangle = \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle = 0 + 0\,.$$

Hence, $\|\varphi\|^2$ is constant and then so is $\|\varphi\|$. □

Theorem 3.35. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure. If there exists a nontrivial parallel spinor $\varphi \in \Gamma(M, \Sigma M)$, then (M, g) is Ricci-flat, i.e.,* $\mathrm{Ric} = 0$.

Proof. Let $\varphi \in \Gamma(M, \Sigma M)$ be nontrivial and parallel. By definition of the curvature tensor $R^{\Sigma M}$, we have

$$R^{\Sigma M}(X, Y)\varphi = 0 \quad \text{for all} \quad X, Y \in \mathcal{V}(M)\,.$$

Fix a point $x \in M$, let $(e_1, \ldots, e_n)$ be an ONB of T_xM and $X \in T_xM$. By Exercise 22 we have

$$0 = \sum_{i=1}^n e_i \cdot R_x^{\Sigma M}(e_i, X)\varphi(x) = \frac{1}{2}\,\mathrm{ric}_x(X) \cdot \varphi(x)\,.$$

The previous lemma assures $\varphi(x) \neq 0$. Hence, $\mathrm{ric}_x(X) = 0$ for all $X \in T_xM$, i.e., $\mathrm{ric}_x = 0$. □

A more general notion than that of a parallel spinor is given in the following definition.

Definition 3.36. *Let (M, g) be a Riemannian spin manifold with a fixed spin structure. A spinor $\varphi \in \Gamma(M, \Sigma M)$ for which there exists a number $\zeta \in \mathbb{C}$ such that*

$$\nabla_X\varphi = \zeta X \cdot \varphi \quad \textit{for all} \quad X \in \mathcal{V}(M)$$

is called a Killing spinor with Killing number ζ.

Remark 3.37. *The defining equation for a Killing spinor is in general well overdetermined. Indeed, if M has dimension n the spinor bundle has rank $2^{\lfloor n/2 \rfloor}$. Hence, locally, $\nabla_X\varphi = \zeta X \cdot \varphi$ is a system of $2^{\lfloor n/2 \rfloor}$ equations in n variables. As we will see in the following propositions, neccessary conditions for Killing spinors to exist are quite restrictive.*

Proposition 3.38. *Let (M,g) be a connected Riemannian spin manifold with a fixed Spin-structure and $\varphi \in \Gamma(M,\Sigma M)$ a Killing spinor with Killing number $\zeta \in \mathbb{C}$. Then*

(i) if φ is nontrivial, then $\varphi(x) \neq 0$ for all $x \in M$,
(ii) $D(\varphi) = -n\zeta\varphi$, i.e., φ is an eigenspinor for the Dirac operator with eigenvalue $-n\zeta$.

Proof. (i): Since we already handled the case of parallel spinors, we can assume $\zeta \neq 0$. Let $\gamma : (-\varepsilon,\varepsilon) \to M$ be any smooth curve and let $\psi : (-\varepsilon,\varepsilon) \ni t \mapsto \varphi(\gamma(t)) \in \Sigma M$. Since φ is a Killing spinor we then have

$$\frac{\nabla}{\mathrm{d}t}\psi(t) = (\nabla_{\gamma'(t)}\varphi)_{\gamma(t)} = \zeta\gamma'(t)\cdot\varphi(\gamma(t)) = \zeta\gamma'(t)\cdot\psi(t)\,,$$

i.e., ψ satisfies a first order ordinary linear differential equation. By uniqueness of solutions of ODEs, $\psi(0) = \varphi(\gamma(0)) = 0$ would imply $\psi \equiv 0$. Since γ was arbitrary, this in turn implies $\varphi \equiv 0$.
(ii): Locally, we have

$$D\varphi = \sum_{i=1}^n e_i\cdot\nabla_{e_i}\varphi = \sum_{i=1}^n e_i\cdot\zeta e_i\cdot\varphi = -n\zeta\varphi\,.$$

□

Definition 3.39. *Let (M,g) be a Riemannian manifold. A vector field $X \in \mathcal{V}(M)$ is a Killing (vector) field if*

$$\mathcal{L}_X g = 0\,,$$

where the Lie-derivative on 2-tensors is given by

$$(\mathcal{L}_X h)(Y,Z) := Xh(Y,Z) - h(\mathcal{L}_X Y,Z) - h(Y,\mathcal{L}_X Z)$$

for all $X,Y,Z \in \mathcal{V}(M)$.

Remark 3.40. *The vector field $X \in \mathcal{V}(M)$ is Killing if and only if*

$$\begin{aligned} 0 &= Xg(Y,Z) - g(\mathcal{L}_X Y,Z) - g(Y,\mathcal{L}_X Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\nabla_X Y,Z) + g(Y,\nabla_X Z) - g(\nabla_X Y - \nabla_Y X,Z) - g(Y,\nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X,Z) + g(Y,\nabla_Z X)\,, \end{aligned}$$

i.e., if and only if $Y \mapsto \nabla_Y X$ is a skew-symmetric endomorphism of the tangent bundle.

Remark 3.41. *Let (M,g) be a Riemannian manifold and assume for simplicity that M is compact. The diffeomorphism group* $\mathrm{Diff}(M)$ *of M is an infinite-dimensional (Fréchet-) Lie group and $\mathcal{V}(M)$ together with the Lie-bracket $[\cdot,\cdot]$ on vector fields is its Lie algebra. This can be seen as follows. Suppose we are given a one-parameter group $t \mapsto \Phi^t$ of diffeomorphisms Φ^t of M with $\Phi^0 = \mathrm{id}_M$. Then $p \mapsto X_p := {}^{\mathrm{d}}\!/\!{}_{\mathrm{d}t}{}_{|t=0}\Phi^t(p)$ clearly is a vector field of M. On the other hand, given any $X \in \mathcal{V}(M)$, then, by compactness, X is complete, i.e., for any starting point $p \in M$ the flow $\Phi^t_X(p)$ exists for all time $t \in \mathbb{R}$. In particular, $t \mapsto \Phi^t_X$ is a one-parameter group of diffeomorphisms with $\Phi^0 = \mathrm{id}_M$.*

Inside $\mathrm{Diff}(M)$ *we have the isometry group*

$$\mathrm{Isom}(M,g) := \{\Phi \in \mathrm{Diff}(M)\,|\,\mathrm{d}\Phi_x : (T_xM, g_x) \to (T_{\Phi(x)}M, g_{\Phi(x)}) \textit{ is an isometry for all } x \in M\}\,.$$

This is a (finite-dimensional) Lie group as in Section 1.1. While for a generic *Riemannian metric g on M the isometry group* $\mathrm{Isom}(M,g)$ *will be trivial, there are Riemannian manifolds whose isometry group has dimension $\geqslant 1$. The most prominent example is of course (S^n, g_{round}) with isometry group $\mathrm{Isom}(S^n,g_{round}) = \mathrm{O}(n+1)$. A noncompact example is the hyperbolic plane $(\mathbb{H}, g_{hyp})$, where $\mathbb{H} = \{(x,y) \in \mathbb{R}^2\,|\,y > 0\}$ and $g_{hyp} = {}^1\!/\!{}_{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$, with isometry group* $\mathrm{Isom}(\mathbb{H},g_{hyp}) = \mathrm{Sl}(2;\mathbb{R})$ *acting by Möbius transformations.*

A Killing field X is a vector field for which the associated flow Φ^t_X is a one-parameter group of isometries *of (M,g), i.e., for each $t \in \mathbb{R}$ the map $M \ni p \mapsto \Phi^t_X(p) \in M$ is an isometry. Thus, the existence of a Killing field $X \in \mathcal{V}(M)$ on a Riemannian manifold (M,g) is equivalent to the assertion that the isometry group* $\mathrm{Isom}(M,g)$ *has positive dimension. Killing fields are sometimes called* infinitesimal isometries.

A typical Killing field on the round sphere can be obtained by differentiating the one-parameter group of rotations around a fixed axis. An example of a Killing field on the hyperbolic plane is $\frac{\partial}{\partial x}$ which corresponds to the one-parameter group of translations along lines parallel to the x-axis.

Proposition 3.42. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure and $\varphi \in \Gamma(M, \Sigma M)$ a Killing spinor with Killing number $\zeta \in \mathbb{R}$. Then the vector field*

$$X := \sum_{j=1}^{n} \mathrm{i}\langle\varphi, e_j \cdot \varphi\rangle e_j \in \mathcal{V}(M)\,,$$

where $(e_1, \ldots, e_n)$ is a local ONB, is a (possibly vanishing) Killing field of (M, g).

Proof. Let $p \in M$ and $(e_1, \ldots, e_n)$ a local ONB in a neighborhood of p with $(\nabla e_j)_p = 0$ for all $j = 1, \ldots, n$. Let $Y \in T_pM$. Then, at p, we have

$$\begin{aligned}
\nabla_Y X &= \mathrm{i} \sum_{j=1}^{n} \big(Y(\langle\varphi, e_j \cdot \varphi\rangle)e_j + \langle\varphi, e_j \cdot \varphi\rangle \nabla_Y e_j\big) \\
&= \mathrm{i} \sum_{j=1}^{n} \big(\langle\nabla_Y \varphi, e_j \cdot \varphi\rangle + \langle\varphi, \nabla_Y (e_j \cdot \varphi)\rangle\big)\, e_j \\
&= \mathrm{i} \sum_{j=1}^{n} \big(\langle\nabla_Y \varphi, e_j \cdot \varphi\rangle + \langle\varphi, \nabla_Y e_j \cdot \varphi\rangle + \langle\varphi, e_j \cdot \nabla_Y \varphi\rangle\big)\, e_j \\
&= \mathrm{i}\zeta \sum_{j=1}^{n} \big(\langle Y \cdot \varphi, e_j \varphi\rangle + \langle\varphi, e_j \cdot Y \cdot \varphi\rangle\big)\, e_j \\
&= \mathrm{i}\zeta \sum_{j=1}^{n} \langle\varphi, e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi\rangle e_j\,,
\end{aligned}$$

so that

$$\begin{aligned}
g(\nabla_Y X, Z) &= \mathrm{i}\zeta \sum_{j=1}^{n} \langle\varphi, e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi\rangle g(e_j, Z) = \mathrm{i}\zeta \sum_{j=1}^{n} \langle\varphi, g(e_j, Z)(e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi)\rangle \\
&= \mathrm{i}\zeta\langle\varphi, Z \cdot Y \cdot \varphi - Y \cdot Z \cdot \varphi\rangle\,,
\end{aligned}$$

which is skew-symmetric in (Y, Z), i.e., $Y \mapsto \nabla_Y X$ is a skew-symmetric endomorphism of the tangent bundle TM. By the last remark, X is a Killing field. □

Proposition 3.43. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure. Assume there exists a Killing spinor $\varphi \in \Gamma(M, \Sigma M)$ with Killing number $\zeta \in \mathbb{C}$. Then we have:*

(i) $\mathrm{ric}(X) = 4(n-1)\zeta^2 X$. *In particular, (M, g) is an Einstein manifold with $\zeta^2 = \frac{1}{4}\frac{\mathrm{scal}}{n(n-1)}$ and $\zeta \in \mathbb{R}$ or $\zeta \in \mathrm{i}\mathbb{R}$.*

(ii) If $\zeta \neq 0$ then (M, g) is locally irreducible, i.e., no point admits a neighborhood U such that $(U, g_{|U})$ is isometric to a Riemmanian product $(V, g_V) \times (W, g_W)$.

Proof. By definition of the curvature tensor we have

$$\begin{aligned}
R^{\Sigma M}(X, Y)\varphi &= \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]}\varphi = \nabla_X(\zeta Y \cdot \varphi) - \nabla_Y(\zeta X \cdot \varphi) - \zeta[X, Y]\varphi \\
&= \zeta\,(\nabla_X Y \cdot \varphi + Y \cdot \nabla_X \varphi - \nabla_Y X \cdot \varphi - X \cdot \nabla_Y \varphi - [X, Y] \cdot \varphi) \\
&= \zeta\,(\nabla_X Y - \nabla_Y X - [X, Y])\,\varphi + \zeta\,(Y \cdot \zeta X \cdot \varphi - X \cdot \zeta Y \cdot \varphi) \\
&= \zeta^2(Y \cdot X - X \cdot Y)\varphi\,.
\end{aligned}$$

Exercise 22 now gives

$$\begin{aligned}
\mathrm{ric}(X) \cdot \varphi &= -2 \sum_{i=1}^{n} e_i \cdot R^{\Sigma M}(X, e_i)\varphi = -2\zeta^2 \sum_{i=1}^{n} e_i \cdot (e_i \cdot X - X \cdot e_i)\varphi = -2\zeta^2 \sum_{i=1}^{n} (e_i^2 \cdot X - e_i \cdot X \cdot e_i)\varphi \\
&= -2\zeta^2 \sum_{i=1}^{n} (e_i^2 \cdot X + e_i^2 \cdot X + 2g(X, e_i)e_i)\varphi = 4(n-1)\zeta^2 X \cdot \varphi\,.
\end{aligned}$$

By Proposition 3.38(i), φ is nowhere zero, which implies $\operatorname{ric}(X) = 4(n-1)\zeta^2 X$, or, equivalently, $\operatorname{Ric}(X,Y) = 4(n-1)\zeta^2 g(X,Y)$. A straightforward calculation yields

$$\operatorname{scal} = \sum_{i=1}^{n} \operatorname{Ric}(e_i, e_i) = \sum_{i=1}^{n} 4(n-1)\zeta^2 g(e_i, e_i) = 4n(n-1)\zeta^2 .$$

To see (ii) assume $U \subseteq M$ is open and that $(U, g_{|U})$ is isometric to the Riemannian product $(V, g_V) \times (W, g_W)$ by an orientation preserving isometry f. We give $(V \times W, g_{V\times W} = g_V \oplus g_W)$ the Spin-structure induced by f so that the spinor bundles over U and $V \times W$ are isomorphic by a vector bundle isomorphism which preseres bundle metrics and covariant derivatives. We now view φ as a spinor on $V \times W$. Let $(x,y) \in V \times W$, $X \in T_xV\backslash\{0\}$, $Y \in T_yW\backslash\{0\}$, so that $X + Y \in T_xV \oplus T_yW \cong T_{(x,y)}V \times W$. Then $R^{V\times W}(X,Y)Z = 0$ for all $Z \in T_xV \oplus T_yW$.

From the above we have on the hand

$$R^{\Sigma(V\times W)}(X,Y)\varphi(x,y) = \frac{1}{4}\sum_{i=1}^{n} e_i \cdot R^{V\times W}(X,Y)e_i \cdot \varphi(x,y) = 0$$

and on the other hand

$$R^{\Sigma(V\times W)}(X,Y)\varphi(x,y) = \zeta^2(Y\cdot X - X\cdot Y)\varphi(x,y) .$$

Since $\zeta \neq 0$ and $g_{V\times W}(X,Y) = 0$ this implies

$$X \cdot Y \cdot \varphi(x,y) = 0 .$$

But Clifford multiplication by a nonzero vector is an isomorphism ($X \cdot X \cdot \varphi(x,y) = -\|X\|^2\varphi(x,y)$), hence $\varphi(x,y) = 0$, which contradicts Proposition 3.38(i). ☐

Corollary 3.44. *Let (M,g) be a connected Riemannian spin manifold with a fixed Spin-structure. Assume there exists a Killing spinor $\varphi \in \Gamma(M, \Sigma M)$ with Killing number $\zeta \neq 0$.*

(i) If ζ is real and (M,g) complete, then M is compact.
(ii) If ζ is imaginary, M is noncompact.

Proof. By the last proposition we have $\operatorname{Ric} = 4(n-1)\zeta^2 g$. If ζ is real, $4(n-1)\zeta^2 > 0$, and Myers' theorem asserts that M is compact.

If ζ is imaginary, we have $\zeta^2 < 0$ and by Proposition 3.38(ii), φ is an eigenspinor of D^2 with eigenvalue $n^2\zeta^2 < 0$. Assuming M is compact implies

$$0 \leqslant (D\varphi, D\varphi) = (D^2\varphi, \varphi) = n^2\zeta^2(\varphi,\varphi) < 0 ,$$

a contradiction. Hence, M must be noncompact. ☐

3.3. **Some Analytic Properties of the Dirac operator.** We recall from Definition 2.12(i)(c) that for any $\mathbb{K}$-vector bundle E of rank k over a smooth manifold M, there exists for any point $x \in M$ an open neighborhood $U \subseteq M$ of x and a local frame $s = (s_1, \dots, s_k) : U \to E^k$, i.e., $(s_1(y), \dots, s_k(y))$ is a basis of E_y for all $y \in U$. Thus, we can express any section $\varphi \in \Gamma(U, E)$ (pointwise) w.r.t. $(s_1, \dots, s_k)$, i.e.,

$$\varphi = \sum_{i=1}^{k} \varphi_i s_i$$

with suitable $\varphi_i \in C^\infty(U, \mathbb{K})$ for all $i = 1, \dots, k$.

Definition 3.45. *Let M be an n-dimensional manifold and E, F two $\mathbb{K}$-vector bundles over M of rank k and l, respectively. A $\mathbb{K}$-linear map $P : \Gamma(M,E) \to \Gamma(M,F)$ is an m-th ($m \in \mathbb{N}_0$) order (linear partial) differential operator if*

- *for all $\varphi \in \Gamma(M,E)$ and for all $x \in M$, $(P\varphi)(x)$ does not depend on the values of φ outside of an arbitrarily small neighborhood of x,*
- *for any (small enough) chart $(U, x = (x^1, \dots, x^n))$ of M, local frames $s = (s_1, \dots, s_k) : U \to E^k$ and $t = (t_1, \dots, t_l) : U \to F^l$, there exists, for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \dots + \alpha_n \leqslant m$, a smooth function*

$$P_\alpha : U \to M(l, k; \mathbb{K})$$

such that for all smooth functions $\varphi_1, \dots, \varphi_k \in C^\infty(U, \mathbb{K})$ *we have*

$$P \sum_{i=1}^{k} \varphi_i s_i = \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{|\alpha| \leqslant m} (P_\alpha)_{ji} \frac{\partial^{|\alpha|} \varphi_i}{\partial x^\alpha} t_j \,,$$

and where we require that for all $y \in U$ *there exists an* α *with* $|\alpha| = m$ *such that*

$$P_\alpha(y) \neq 0 \,.$$

We denote the space of all m-th order linear partial differential operators from E to F by $\mathscr{D}^m(M; E; F)$ *and write* $\mathscr{D}^m(M; E)$ *for* $\mathscr{D}^m(M; E; E)$.

Example 3.46. *Let M be a smooth manifold and* E, F *two* $\mathbb{K}$*-vector bundles over M.*

(i) *Any connection* $\nabla : \Gamma(E; M) \to \Gamma(M, T^*M \otimes E)$ *is a 1-st order linear partial differential operator.*
(ii) *Any vector bundle homomorphism* $\Phi : E \to F$, *extended to a map*

$$\Phi : \Gamma(M, E) \ni s \mapsto (x \mapsto \Phi_{|E_x} s(x)) \in \Gamma(M, F) \,,$$

is a 0-th order linear partial differential operator.
(iii) *Any Bochner Laplacian* $\Delta^E : \Gamma(M, E) \to \Gamma(M, F)$ *associated with a connection* ∇^E *in E is a 2-nd order linear partial differential operator.*

For the next definition, recall the m-fold symmetric tensor product $V^{\odot m}$ of a $\mathbb{K}$-vector spaces V, which is the subspace of $V^{\otimes m}$ which is invariant w.r.t. the linear maps

$$V^{\otimes m} \ni v_1 \otimes \dots \otimes v_m \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(m)} \in V^{\otimes m} \,, \quad \sigma \in S_m \,.$$

Definition 3.47. *Let M be a smooth manifold,* E, F *two* $\mathbb{K}$*-vector bundles over M and* $P \in \mathscr{D}^m(M; E; F)$.

(i) *The symbol of P is the vector bundle homomorphism* $\sigma(P) : (T^*M)^{\odot m} \otimes E \to F$ *defined by*

$$\sigma(P) : (T_x^*M)^{\odot m} \otimes E_x \ni \xi^{\odot m} \otimes e \mapsto \frac{1}{m!} P(f^m s)(x) \in F_x \,,$$

where $s \in \Gamma(M, E)$ *is any extension of* $e \in E_x$ *and* $f \in C^\infty(M)$ *is such that* $f(x) = 0$ *and* $\mathrm{d}f_x = \xi \in T_x^*M$.
(ii) *We call P elliptic if for all* $x \in M$ *and* $\xi \in T_x^*M \backslash \{0\}$, *the map*

$$\sigma(P)_\xi = \sigma(P)(\xi^{\odot m} \otimes \cdot) : E_x \to F_x$$

is an isomorphism.
(iii) *If we are given a Riemannian metric g on M, and if* $E = F$ *and P is of second order (m=2), we call P a generalized Laplacian or an operator of Laplace type if*

$$\sigma(P)_\xi = -\|\xi\|_g^2 \cdot \mathrm{id}_{E_x}$$

for all $x \in M$ *and* $\xi \in T_x^*M$.

Lemma 3.48. *The symbol* $\sigma(P)$ *of* $P \in \mathscr{D}^m(M; E; F)$ *is well-defined. In particular, it is independent of s and f. More precisely, in charts as in Definition 3.45,* $\sigma(P)_\xi$ *is given by*

$$\sum_{|\alpha| = m} \xi^\alpha P_\alpha \,,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$ *and* $\xi = \sum_j \xi_j \mathrm{d}x_j$.

Proof. Exercise. □

Lemma 3.49. *If* $P \in \mathscr{D}^k(M; E; F)$ *and* $Q \in \mathscr{D}^l(M; F; G)$, *then* $Q \circ P \in \mathscr{D}^{k+l}(M; E; G)$ *and* $\sigma(Q \circ P)_\xi = \sigma(Q)_\xi \circ \sigma(P)_\xi$.

Proof. Straightforward using the previous lemma. □

Example 3.50. *Let* (M, g) *be a Riemannian manifold.*

(i) *Suppose (M,g) is spin and fix a Spin-structure. Consider the Dirac operator $D : \Gamma(M,\Sigma M) \to \Gamma(M,\Sigma M)$, which was defined as the superposition $\mu \circ (\sharp \otimes \mathrm{id}_{\Sigma M}) \circ \nabla$ of the spinor connection $\nabla : \Gamma(M,\Sigma M) \to \Gamma(M, T^*M \otimes \Sigma M)$, the tensor product $\sharp \otimes \mathrm{id} : T^*M \otimes \Sigma M \to TM \otimes \Sigma M$ of the musical isomorphism and the identity of ΣM and Clifford multiplication $\mu : TM \otimes \Sigma M \to \Sigma M$. By Examples 3.46(i) and (ii), $\nabla \in \mathscr{D}^1(M;\Sigma M; T^*M \otimes \Sigma M)$, $\sharp \otimes \mathrm{id} \in \mathscr{D}^0(M; T^*M \otimes \Sigma M; TM \otimes \Sigma M)$ and $\mu \in \mathscr{D}^0(M; TM \otimes \Sigma M; \Sigma M)$ so that by the last lemma we have $D \in \mathscr{D}^1(M;\Sigma M)$ and $D^2 \in \mathscr{D}^2(M;\Sigma M)$. The symbol of D is given by*

$$\sigma(D)_\xi(e) = \frac{1}{1!} D(f^1\varphi)(x) = \operatorname{grad} f_x \cdot \varphi(x) + f(x) D\varphi(x) = \operatorname{grad} f_x \cdot e = \xi^\sharp \cdot e\,,$$

where f, and φ are as in Definition 3.47. Since Clifford multiplication by a nonzero vector is a linear isomorphism, D is an elliptic operator. Moreover,

$$\sigma(D^2)_\xi(e) = \sigma(D)_\xi \circ \sigma(D)_\xi(e) = \xi^\sharp \cdot \xi^\sharp \cdot e = -\|\xi\|^2 e\,,$$

i.e., D^2 is a Laplace type operator.

(ii) *We consider the Laplace-Beltrami operator $\Delta : C^\infty(M) = \Gamma(M, M \times \mathbb{R}) \to C^\infty(M)$, $\nabla f = -\operatorname{div} \operatorname{grad} f$. The gradient of a function f is given by $\operatorname{grad} f = (\mathrm{d}f)^\sharp$, a superposition of the differential and the musical isomorphism. We argue as in the last example to see that $\operatorname{grad} \in \mathscr{D}^1(M;\mathbb{R};TM)$. From Definition 3.19 and Example 3.46(i), we see that $\operatorname{div} \in \mathscr{D}^1(M;TM;\mathbb{R})$ so that, by the last lemma, we have $\Delta = -\operatorname{div} \circ \operatorname{grad} \in \mathscr{D}^2(M;\mathbb{R})$. To compute the symbol of Δ, let $x \in M$, $f, h \in C^\infty(M)$ with $f(x) = 0$, $\mathrm{d}f_x = \xi \in T_x^*M$, $h(x) = 1$, $X \in T_xM$ as well as $\widetilde{X} \in \mathcal{V}(M)$ with $\widetilde{X}_x = X$. Then, by Exercise 20 we have*

$$\begin{aligned}
\sigma(\operatorname{grad})_\xi(1) &= \frac{1}{1!}\operatorname{grad}(f^1h)(x) = (\mathrm{d}(fh)_x)^\sharp = ((\mathrm{d}f_x)h(x) + f(x)\mathrm{d}h_x)^\sharp = \operatorname{grad} f_x h(x) + f(x) \operatorname{grad} h_x \\
&= \xi^\sharp\,, \\
\sigma(\operatorname{div})_\xi(X) &= \frac{1}{1!}\operatorname{div}(f^1\widetilde{X})(x) = g_x(\operatorname{grad} f_x, X) + f(x)\operatorname{div} \widetilde{X} = g(\operatorname{grad} f_x, X) = \mathrm{d}f_x(X) = \xi(X)\,,
\end{aligned}$$

so that

$$\sigma(\Delta)_\xi(1) = -\sigma(\operatorname{div})_\xi \circ \sigma(\operatorname{grad})_\xi(1) = -\xi(\xi^\sharp) = -\|\xi\|^2\,.$$

This is the justification for the name generalized Laplacian.

Before plunging into spectral theory, let us recall a few facts from functional analysis which can be looked up in virtually any textbok covering densely defined operators in Hilbert space.

Let $\mathscr{H}$ be a seperable Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $A : \mathscr{H} \supseteq D(A) \to \mathscr{H}$ a linear operator defined on a subspace $D(A) \subseteq \mathscr{H}$, called the domain of A. We denote the inner product of $\mathscr{H}$ by $(\cdot,\cdot)$, which we assume to be $\mathbb{C}$-linear in the first slot and $\mathbb{C}$-antilinear in the second one.

- We call a linear operator $B : \mathscr{H} \supseteq D(B) \to \mathscr{H}$ an extension of A, denoted $A \subseteq B$, if $D(A) \subseteq D(B)$ and $Av = Bv$ for all $v \in D(A)$. We also write $A = B$ if $A \subseteq B$ and $B \subseteq A$.
- We call A densely defined if $D(A) \subseteq \mathscr{H}$ is dense.
- A is hermitian if it is formally selfadjoint to itself, i.e., $(Av,w) = (v,Aw)$ for all $v,w \in D(A)$.
- If A is densely defined and hermitian, it is symmetric.
- The operator A is closed if the graph $\Gamma(A) = \{(x,Tx) \in \mathscr{H} \times \mathscr{H} \mid x \in \mathscr{H}\} \subseteq \mathscr{H} \times \mathscr{H}$ is closed.
- A is closable if the closure $\overline{\Gamma(A)}$ of the graph $\Gamma(A)$ of A is the graph of an operator $\overline{A}$. We call $\overline{A}$ the closure of A.
- If A is densely defined, the adjoint A^* of A is the operator $A^* : \mathscr{H} \supseteq D(A^*) \to \mathscr{H}$ with

$$\begin{aligned}
D(A^*) &:= \{w \in \mathscr{H} \mid \ell : D(A) \ni v \mapsto (Av,w) \in \mathbb{C} \text{ is a continuous functional}\}\,, \\
A^*w &:= z\,,
\end{aligned}$$

  where $\overline{\ell} =: (\cdot, z)$ (Fréchet-Riesz) and $\overline{\ell} : \mathscr{H} \to \mathbb{C}$ is the continuous extension of ℓ to $\mathscr{H}$.
- We call A selfadjoint if $A^* = A$.
- If A is symmetric, then:
  - A^* is densely defined and closed.
  - A is closable and $\overline{A} = A^{**} \subseteq A^*$.
  - For any selfadjoint extension B of A, we have

$$A \subseteq \overline{A} \subseteq B = B^* \subseteq A^*\,.$$

  - We call A essentially selfadjoint if A posses a *unique* selfadjoint extension. We then have neccessarily $\overline{A} = A^*$.
- The resolvent set of A is

$$\rho(A) := \left\{ z \in \mathbb{K} \mid (A - z\,\mathrm{id}) \text{ is bijective as map from } D(A) \to \mathscr{H} \text{ and } (T - z\,\mathrm{id})^{-1} \text{ is continuous} \right\}$$

  and the spectrum of A

$$\mathrm{spec}(A) := \mathbb{K} \backslash \rho(A) \,.$$

  For a selfadjoint A we always have $\mathrm{spec}(A) \subseteq \mathbb{R}$.
- An eigenvalue of A is a number $\lambda \in \mathrm{spec}(A)$ such that $(A - \lambda\,\mathrm{id}) : D(A) \to \mathscr{H}$ is not injective. We call $\dim\ker(A - \lambda\,\mathrm{id})$ the multiplicity of λ and any $v \in \ker(A - \lambda\,\mathrm{id}) \backslash \{0\}$ an Eigenvector for the eigenvalue λ.
- If A is selfadjoint and nonnegative , i.e., $(Av, v) \geqslant 0$ for all $v \in D(A)$, then $\mathrm{spec}(A) \subseteq [0, \infty)$.

Definition and Remarks 3.51. *Let (M, g) be a Riemannian manifold and $(E, \pi_E; V)$ a $\mathbb{K}$-vector bundle over M, equipped with a bundle metric.*

(i) A measurable section of E is a measurable[1] map $s : M \to E$ with $\pi_E \circ s = \mathrm{id}_M$. We call two measurable sections $s, s' : M \to E$ equivalent if they agree μ_g-almost everywhere and denote the corresponding equivalence class by $[s]$.

(ii) The space of L^2-sections of E is

$$\Gamma_{L^2}(M; E) := \{[s] \mid [s] \text{ is an equivalence class of measurable sections of } E \text{ with } |[s]|_{L^2} < \infty\} \,,$$

where we extended the the L^2-inner product $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$ and its norm $|\cdot|$, initially defined on $\Gamma_c(M; E)$, to all equivalence classes of measurable sections, noting that it is independent of the representatives due to a.e.-equivalence of the sections. $(\Gamma_{L^2}(M; E), (\cdot,\cdot)_{L^2})$ is a complete, seperable $\mathbb{K}$-Hilbert space. In case E is the trivial $\mathbb{K}$-vector bundle $E = M \times \mathbb{K}$, the corresponding space of L^2-sections is just $L^2_{\mathbb{K}}(M)$, the space of (equivalence classes of) $\mathbb{K}$-valued square-integrable functions.

(iii) We view $\Gamma_c(M; E)$ as a subspace of $\Gamma_{L^2}(M; E)$ via the inclusion

$$\iota : \Gamma_c(M; E) \ni f \mapsto [f] \in \Gamma_{L^2}(M; E)$$

and note that, using a partition of unity and the usual approximation argument, it is not hard to see that $\Gamma_c(M; E)$ is dense in $\Gamma_{L^2}(M; E)$.

(iv) Assuming (M, g) to be a Riemannian spin manifold with a fixed Spin-structure and using Proposition 3.22, we see that the Dirac operator D defined on *$\Gamma_c(M; \Sigma M)$ is a symmetric operator in $\Gamma_{L^2}(M; \Sigma M)$.*

(v) If the vector bundle E over M comes equipped with a bundle metric and a metric connection ∇^E, then by Proposition 3.29 the associated Bochner-Laplacian Δ^E defined on *$\Gamma_c(M; E)$ is a symmetric operator in $\Gamma_{L^2}(M; E)$. Similarly, the Laplace-Beltrami operator $\Delta = -\mathrm{div}\,\mathrm{grad}$ defined on $C^\infty_c(M; \mathbb{K})$ is a symmetric in $L^2_{\mathbb{K}}(M)$ by Exercise 20.*

(vi) More generally, any $P \in \mathscr{D}^m(M; E)$, viewed as an operator in $\Gamma_{L^2}(M; E)$ with domain $D(P) = \Gamma_c(M; E)$ is a densely defined operator.

Theorem 3.52. *Let (M, g) be a* complete *Riemannian spin manifold with a fixed Spin-structure. Then the Dirac operator D, initially defined on $\Gamma_c(M; \Sigma M)$, is essentially selfadjoint in $\Gamma_{L^2}(M; \Sigma M)$. Denoting the closure of D again by D, we have moreoever*

$$\ker D = \ker D^2 \,.$$

Proof. The original proof by J. A. Wolf is contained in [Fr00]. A considerably shorter proof relying on distribution theory is given in [LM89]. □

Remark 3.53. *The assumption that the manifold (M, g) is complete can i.g. not be dropped. In fact, there are noncomplete manifolds for which the Dirac operator does not posses any selfadjoint extension. This is in stark contrast to the following theorem.*

Theorem 3.54. *Let (M, g) be a Riemannian manifold and E a $\mathbb{K}$-vector bundle, equipped with a bundle metric and a metric connection ∇^E. Let Δ be either the Bochner-Laplacian associated with ∇^E, seen as an operator in $\Gamma_{L^2}(M; E)$ with dense domain $\Gamma_c(M; E)$, or the Laplace-Beltra operator in $L^2_{\mathbb{K}}(M)$ with dense domain $C^\infty_c(M; \mathbb{K})$. Then Δ has a unique,*

[1] w.r.t. the Borel σ-algebras $\mathcal{B}(M)$ and $\mathcal{B}(E)$

minimal[2] *selfadjoint extension Δ_F, called the* Friedrichs extension *. In case that the manifold is complete, Δ is essentially selfadjoint and the Friedrichs extension conincides with the closure of Δ.*

We are interested mainly in the situation where the manifold M is closed , i.e., compact and without boundary (we have not dealt with manifolds with boundary so far and we will not do so). Here, the spectral theory of any essentially selfadjoint elliptic differential operator is completely understood. In particular, we know the spectral situation of the Dirac operator, the Laplace-Beltrami operator, and any Bochner-Laplacian over a closed Riemannian manifold.

Theorem 3.55 (see, e.g., [LM89, Chapter III, § 5, Theorem 5.8]). *Let (M, g) be a closed Riemannian manifold and E a $\mathbb{K}$-vector bundle over M, equipped with a bundle metric. Assume $P \in \mathscr{D}^m(M; E)$ is elliptic and essentially selfadjoint, and denote the closure of P again with P. Then:*

(i) The spectrum $\operatorname{spec}(P)$ *of P is discrete and consists only of eigenvalues. Each eigenvalue has finite multiplicity.*
(ii) There exists a complete orthonormal system $(\varphi_i)_{i\in I}$ of $\Gamma_{L^2}(M; E)$ consisting of smooth eigensections of P.

Example 3.56. *Let $M = S^1 \cong [0, 2\pi]/\{0, 2\pi\}$ with its metric coming from the embedding $S^1 \subseteq \mathbb{C} \cong \mathbb{R}^2$ and the trivial Spin-structure (see Example 3.5) $P_1 = S^1 \times \mathbb{Z}_2$. The Clifford algebra over $\mathbb{C}$ is given by $\mathbb{C}\ell_1 = \{a \cdot 1 + b \cdot e_1 \mid a, b \in \mathbb{C}\}$, which is isomorphic to the product algebra $\mathbb{C} \oplus \mathbb{C}$, the isomorphism given by $\mathbb{C}\ell_1 \ni a \cdot 1 + b \cdot e_1 \mapsto (a + \mathrm{i}b, a - \mathrm{i}b) \in \mathbb{C} \oplus \mathbb{C}$. An irreducible representation is thus an algebra homomorphism $\rho : \mathbb{C}\ell_1 \to \mathsf{End}(\mathbb{C}) \cong \mathbb{C}$, and its restriction to $\mathrm{Spin}(1) = \mathbb{Z}_2 = \{\pm 1\}$ is then simply given by multiplication, i.e., $\kappa_1 : \mathrm{Spin}(1) = \mathbb{Z}_2 \ni g \mapsto (\mathbb{C} \ni x \mapsto gx \in \mathbb{C}) \in \mathsf{End}(\Sigma_1)$. Hence, $\Sigma S^1 = P_1 \times_{\kappa_1} \Sigma_1 = S^1 \times \mathbb{Z}_2 \times_{\kappa_1} \mathbb{C} = S^1 \times \mathbb{C}$, the trivial vector bundle with fibre $\mathbb{C}$ and its sections are just $\mathbb{C}$-valued functions. Clifford multiplication by e_1 satisfies $e_1^2 = -1$, but $\Sigma_1 = \mathbb{C}$, so e_1 is either $+\mathrm{i}$ or $-\mathrm{i}$. Recall that in odd dimensions, we always choose Clifford multiplication such that multiplication by the volume element acts as the identity. Hence, $1 = \omega_1 = \mathrm{i}^{\frac{1+1}{2}} e_1 = \mathrm{i}e_1$, which means $e_1 = -\mathrm{i}$. It follows that, w.r.t. the coordinate chart $(0, 2\pi) \ni t \mapsto \mathrm{e}^{\mathrm{i}t} \in S^1$, the Dirac operator is given by $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$. For $k \in \mathbb{Z}$, we consider the smooth 2π-periodic function $\varphi_k : (0, 2\pi) \ni t \mapsto \mathrm{e}^{\mathrm{i}kt} \in \mathbb{C}$. We have*

$$D\varphi_k(t) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}kt} = -\mathrm{i}\mathrm{i}k\mathrm{e}^{\mathrm{i}kt} = k\varphi_k(t)\,.$$

Moreover, from Fourier analysis we know that $(\varphi_k)_{k\in\mathbb{Z}}$ is a complete orthonormal system of $L^2_{\mathbb{C}}(S^1)$. By the last theorem, part (ii), we have found all the eigenvalues of D, namely, $\operatorname{spec}(D) = \mathbb{Z}$*, and each $k \in \mathbb{Z}$ has multiplicity 1.*

Exercise 3.57. *Compute the spectrum of the Dirac operator on S^1 equipped with the nontrivial Spin-structure. Compare this with the above example.*

Next, we are going to compute the spectrum of the Dirac operator on the round sphere $(S^n, g_{\mathrm{round}})$, $n \geqslant 2$. This was first done by S. Sulanke in her Ph.D.-thesis. We follow an approach taken by C. Bär [Ba96] for which we will first need the spectrum of the Laplace-Beltrami operator on the sphere, but which has the advantage that it is elementary.

Theorem 3.58. *The eigenvalues of the Laplace-Beltrami operator Δ on (S^n, g_{round}) are $k(n + k - 1)$, $k \in \mathbb{N}_0$, with corresponding multiplicity $m_k := \binom{n+k-1}{k}\frac{n+2k-1}{n+k-1}$.*

Sketch of the proof. We denote by Δ^{S^n} the Laplacian on $(S^n, g_{\mathrm{round}})$ and by $\Delta^{\mathbb{R}^{n+1}}$ the one on $(\mathbb{R}^{n+1}, g_{Eukl})$.

Step 1: One proves that for each $f \in C^\infty(\mathbb{R}^{n+1})$, one has

$$(\Delta^{\mathbb{R}^{n+1}} f)_{|S^n} = \Delta^{S^n}(f_{|S^n}) - N(Nf) - nNf\,,$$

where $N \in \mathcal{V}(S^n)$ is the outward-pointing unit normal vector field to S^n, $N : S^n \ni x \mapsto x \in (T_x S^n)^\perp \subseteq \mathbb{R}^{n+1}$. One does so by using

$$\nabla^{\mathbb{R}^{n+1}}_X Y = \nabla^{S^n}_X Y - \langle X, Y\rangle N$$

and

$$\Delta = -\mathrm{tr}_g \nabla df\,.$$

Step 2: Let $f \in C^\infty(\mathbb{R}^{n+1})$ be a homogeneous, harmonic ($\Delta^{\mathbb{R}^{n+1}} f = 0$) polynomial of degree k, $k \in \mathbb{N}_0$. By the above formula we have

$$\Delta^{S^n}(f_{|S^n}) = (\Delta^{\mathbb{R}^{n+1}} f)_{|S^n} + N(Nf) + nNf = 0 + k(k-1) + nk = k(n + k - 1)\,.$$

[2] If A is any selfadjoint extension of Δ, then $\Delta_F \subseteq A$

Step 3: Denote by P_k the space of homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$ and let $H_k \subseteq P_k$ be the subspace of harmonic polynomials. We let $\mathscr{P}_k := \{f_{|S^n} \mid f \in P_k\}$ and $\mathscr{H}_k := \{f_{|S^n} \mid f \in H_k\}$, the sets obtained by restricting the elements of P_k resp. H_k to S^n. Denoting by r the radial coordinate on $\mathbb{R}^{n+1}$, one proves the decompositions

$$P_k = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} r^{2j} H_{k-2j}\,,$$

$$\mathscr{P}_k = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \mathscr{H}_{k-2j}\,,$$

which shows that $\dim \mathscr{H}_k = \dim \mathscr{P}_k - \dim \mathscr{P}_{k-2} = \binom{n+k}{k} - \binom{n+k-2}{k-2} = \binom{n+k-1}{k}\frac{n+2k-1}{n+k-1}$.

Step 4: Since S^n is compact, $\bigoplus_{k\geqslant 0} \mathscr{P}_k$ is dense in $C^0(S^n)$ (w.r.t. uniform convergence), which in turn is dense in $L^2(S^n)$. Each $\mathscr{P}_k$ is a direct sum of Eigenspaces of Δ^{S^n} by the previous two steps. Hence, we have accounted for all the eigenvalues of Δ^{S^n}. □

We will now compute the spectrum of the Dirac operator on the round sphere. Recall the Spin-structure on (S^n, g_{round}) that we constructued in Example 3.4 and recall that, because the sphere is *simply-connected*, it is the only Spin-structure on S^n.

On the sphere, we consider for $\zeta \in \{\pm 1/2\}$ the connection $\nabla^\zeta : \Gamma(M, \Sigma M) \to \Gamma(M, T^*M \otimes \Sigma M)$ defined by

$$\nabla^\zeta_X \varphi := \nabla_X \varphi - \zeta X \cdot \varphi\,. \tag{3.7}$$

We also consider the Bochner-Laplacian Δ^ζ associated with ∇^ζ.

Lemma 3.59. *One has the Weitzenböck formula*

$$(D+\zeta)^2 = \Delta^\zeta + \frac{1}{4}(n-1)^2\,.$$

Proof. Let $p \in S^n$ and $(e_1, \ldots, e_n)$ a local ONB around p with $(\nabla e_j)_p = 0$ for all $j = 1, \ldots, n$. At p we obtain

$$\begin{aligned}
(D+\zeta)^2\varphi - \Delta^\zeta \varphi &= \left(\sum_{i=1}^n e_i \cdot \nabla_{e_i} + \zeta\right)\left(\sum_{j=1}^n e_j \cdot \nabla_{e_j}\varphi + \zeta\varphi\right) - \sum_{i=1}^n \nabla^\zeta_{e_i}\nabla^\zeta_{e_i}\varphi \\
&= \sum_{i,j=1}^n e_i \cdot e_j \cdot \nabla_{e_i}\nabla_{e_j}\varphi + 2\zeta D\varphi + \zeta^2\varphi + \sum_{i=1}^n \left(\nabla_{e_i} - \zeta e_i\right)\left(\nabla_{e_i}\varphi - \zeta e_i \cdot \varphi\right) \\
&= -\sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}\varphi + \sum_{1\leqslant i<j\leqslant n} e_i \cdot e_j R^{\Sigma S^n}(e_i, e_j)\varphi + 2\zeta D\varphi + \frac{1}{4}\varphi + \sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}\varphi - 2\zeta D\varphi - \frac{1}{4}n\varphi\,.
\end{aligned}$$

Since the round sphere has constant sectional curvature 1, the Riemannian curvature tensor satisfies $R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$, which implies by Proposition 3.27 that $\mathrm{R}^{\Sigma S^n}(X,Y) = 1/4(Y \cdot X - X \cdot Y)$. Thus, the above is equal to

$$\begin{aligned}
&\frac{1}{4}\sum_{1\leqslant i<j\leqslant n} e_i \cdot e_j \cdot (e_j \cdot e_i - e_i \cdot e_j) \cdot \varphi - \frac{1}{4}(n-1)\varphi \\
&= \frac{1}{4}n(n-1)\varphi - \frac{1}{4}(n-1)\varphi \\
&= \frac{1}{4}(n-1)^2\varphi\,.
\end{aligned}$$

□

Recall from Exercise 23(b) that $R^{\nabla^\zeta} \equiv 0$. Together with the fact that S^n is *simply-connected*, this implies that the spinor bundle ΣS^n can be trivialized by ∇^ζ-parallel spinors, i.e., there exist $2^{\lfloor n/2 \rfloor}$ pointwise linearly independent Killing-spinors $\psi_1, \ldots, \psi_{2^{\lfloor n/2 \rfloor}} \in \Gamma(S^n, \Sigma S^n)$ with Killing number ζ.

Denote the eigenvalues of the Laplace-Beltrami operator Δ on S^n by λ_k, $k \in \mathbb{N}_0$, where we enumerate the eigenvalues according to their finite multiplicity, and let $\{f_k\}_{k\in\mathbb{N}_0}$ be a corresponding complete orthogonal system of Eigenfunctions of Δ for $L^2_{\mathbb{R}}(S^n)$. Then $\{f_k\psi_l\}_{k\in\mathbb{N}_0}^{l=1,\ldots,2^{\lfloor n/2 \rfloor}}$ is a complete orthogonal system of $\Gamma_{L^2}(S^n, \Sigma S^n)$.

Lemma 3.60. *We have*

$$(D+\zeta)^2(f_k\psi_l) = \left(\lambda_k + \tfrac{(n-1)^2}{4}\right) f_k\psi_l$$

for all $k \in \mathbb{N}_0$, $l = 1,\dots,2^{\lfloor n/2\rfloor}$. In particular, the eigenvalues of $(D+\zeta)^2$ are $k(n+k-1) + (n-1)^2/4$, $k \in \mathbb{N}_0$, with corresponding multiplicity $2^{\lfloor n/2\rfloor} m_k$.

Proof. By the last lemma, we have

$$(D+\zeta)^2(f_k\psi_l) = \left(\Delta^{\zeta+\frac{1}{4}(n-1)^2}\right)(f_k\psi_l) = \Delta^{\zeta}(f_k\psi_l) + \frac{1}{4}(n-1)^2 f_k\psi_l\,.$$

Since ψ_l is ∇^ζ-parallel, it is Δ^ζ-harmonic, which means

$$\Delta^\zeta(f_k\psi_l) = (\Delta f_k)\psi_l = \lambda_k f_k \psi_l\,.$$

□

Theorem 3.61. *The eigenvalues of the Dirac operator D on the round sphere S^n, $n \geqslant 2$, are $\pm\left(\frac{n}{2}+k\right)$, $k \in \mathbb{N}_0$, with corresponding multiplicity $2^{\lfloor n/2\rfloor}\binom{n+k-1}{k}$.*

Proof. We begin our proof with a general remark. If an operator A satisfies $A^2u = \nu^2 u$ for some number ν and a nonzero vector u, then the vectors $v^\pm := \pm\nu u + Au$ satisfy

$$Av^\pm = \pm\nu Au + A^2u = \pm\nu Au + \nu^2 u = \nu(\pm Au + \nu u) = \pm\nu v^\pm\,,$$

i.e., if $v^\pm$ is nonzero, it is an Eigenvector of A to the eigenvalue $\pm\nu$.

In the case at hand, $A = D+\zeta$, $\nu = -\zeta(n-1)$ and $u = f_0\psi_l$ for some $l \in \{1,\dots,2^{\lfloor n/2\rfloor}\}$. Since the first eigenvalue λ_0 of Δ is always 0, we can assume $f_0 \equiv 1$[3]. Then

$$v^+ = -\zeta(n-1)\psi_l + (D+\zeta)\psi_l = -\zeta(n-1)\psi_l - n\zeta\psi_l + \zeta\psi_l = -2\zeta(n-1)\psi_l \neq 0\,.$$

Thus, $-\zeta(n-1)$ is an eigenvalue of $D+\zeta$ and since we can choose l freely its multiplicity is at least $2^{\lfloor n/2\rfloor}$. Because the multiplicity of the eigenvalue $(n-1)^2/4$ of $(D+\zeta)^2$ is exactly $2^{\lfloor n/2\rfloor}$, the eigenvalue $-\zeta(n-1)$ of $D+\zeta$ has multiplicity precisely $2^{\lfloor n/2\rfloor}$. Expressed differently, the Dirac operator has the two eigenvalues $\pm n/2$, each with multiplicity $2^{\lfloor n/2\rfloor}$.

Let us come to the case $u = f_k\psi_l$ with $k \geqslant 1$. Then

$$\nu = \sqrt{k(n+k-1) + \frac{1}{4}(n-1)^2} = k + \frac{n-1}{2}\,,$$

meaning that D also has the eigenvalues $-\zeta \pm (k + (n-1)/2)$, $k \in \mathbb{N}$. It remains to determine the multiplicities of these eigenvalues. Recall that we may choose $\zeta = -1/2$ or $\zeta = 1/2$ and we will start with $\zeta = -1/2$. We introduce the notation

$$\begin{aligned}\nu_k^+ &= \frac{n}{2} + k\,, \quad k \in \mathbb{N}_0\,,\\ \nu_{-k}^+ &= 1 - \frac{n}{2} - k\,, \quad k \in \mathbb{N}\,.\end{aligned}$$

We already determined the multiplicity of ν_0^+, namely $m(\nu_0^+) = 2^{\lfloor n/2\rfloor}$. From the last lemma, we know that $m(\nu_k^+) + m(\nu_{-k}^+) = 2^{\lfloor n/2\rfloor} m_k$.

For $\zeta = +1/2$ we use the notation

$$\begin{aligned}\nu_k^- &= -\frac{n}{2} - k\,, \quad k \in \mathbb{N}_0\,,\\ \nu_{-k}^- &= -1 + \frac{n}{2} + k\,, \quad k \in \mathbb{N}\,,\end{aligned}$$

for which we have analogously to the above $m(\nu_0^-) = 2^{\lfloor n/2\rfloor}$ and $m(\nu_k^-) + m(\nu_{-k}^-) = 2^{\lfloor n/2\rfloor} m_k$.

We will now show that $m(\nu_k^+) = m(\nu_k^-) = 2^{\lfloor n/2\rfloor}\binom{n+k-1}{k}$ for all $k \in \mathbb{N}_0$.

[3] On a compact boundaryless manifold, the first eigenvalue of Δ is always zero, has multiplicity one, and the corresponding eigenspace is spanned by the constant functions.

For $k = 0$ we have already shown this. Let us assume that the statement is true for some $k \in \mathbb{N}_0$. Then

$$\begin{aligned} m(v^{\pm}_{k+1}) &= 2^{\lfloor n/2 \rfloor} m_{k+1} - m(v^{\pm}_{-(k+1)}) = 2^{\lfloor n/2 \rfloor} m_{k+1} - m(v^{\mp}_{k}) \\ &= 2^{\lfloor n/2 \rfloor} \left(\binom{n+k}{k+1} \frac{n+2k+1}{n+k} - \binom{n+k-1}{k} \right) \\ &= 2^{\lfloor n/2 \rfloor} \binom{n+k}{k+1}, \end{aligned}$$

which was to be shown. □

Remark 3.62. *Our next goal is an eigenvalue estimate for the Dirac operator. Since the spinor Laplacian is a nonnegative operator, the Lichnerowicz formula tells us that any eigenvalue λ of the Dirac operator on a closed Riemannian manifold (M, g) satisfies $\lambda^2 \geqslant \frac{\mathrm{scal}_0}{4}$, where $\mathrm{scal}_0 := \inf_{x \in M} \mathrm{scal}(x)$. Indeed, let λ be an eigenvalue of D with a corresponding L^2-normalized smooth eigenspinor $\varphi \in \Gamma(M, \Sigma M)$. On the one hand, we have*

$$(D^2\varphi, \varphi) = \lambda^2(\varphi, \varphi) = \lambda^2 ,$$

and on the other hand

$$(D^2\varphi, \varphi) = (\Delta\varphi, \varphi) + \left(\tfrac{1}{4}\,\mathrm{scal}\,\varphi, \varphi\right) \geqslant \tfrac{1}{4}\,\mathrm{scal}_0(\varphi, \varphi) = \tfrac{1}{4}\,\mathrm{scal}_0 .$$

As the next theorem shows, this inequality is not sharp and we can do better.

Theorem 3.63 (Friedich's inequality)**.** *Let (M^n, g) be closed Riemannian spin manifold with fixed Spin-structure. Then every eigenvalue λ of the Dirac operator D satisfies*

$$\lambda^2 \geqslant \frac{n}{n-1} \frac{\mathrm{scal}_0}{4} .$$

Moreover, if $\lambda = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}\,\mathrm{scal}_0}$ is an eigenvalue of the Dirac operator with corresponding eigenspinor φ, then φ is a Killing spinor with Killing number $\mp\frac{1}{2}\sqrt{\frac{1}{n(n-1)}\,\mathrm{scal}_0}$. In particular, the scalar curvature is constant.

Remark 3.64. *Friedrich's inequality is sharp. Indeed, equality is attained on, e.g., the sphere where we have $\mathrm{scal}_0 = \mathrm{scal} = n(n-1)$.*

Proof of Theorem 3.63. Recall the twisted connection ∇^{ζ} from (3.7). For a spinor $\varphi \in \Gamma(M, \Sigma M)$ we have

$$\begin{aligned} \langle \nabla^{-\zeta}\varphi, \nabla^{-\zeta}\varphi \rangle &= \sum_{j=1}^{n} \langle \nabla^{-\zeta}_{e_j}\varphi, \nabla^{-\zeta}_{e_j}\varphi \rangle = \sum_{j=1}^{n} \langle \nabla_{e_j}\varphi + \zeta e_j \cdot \varphi, \nabla_{e_j}\varphi + \zeta e_j \cdot \varphi \rangle \\ &= \sum_{j=1}^{n} \Big(\langle \nabla_{e_j}\varphi, \nabla_{e_j}\varphi \rangle + \zeta\langle e_j \cdot \varphi, \nabla_{e_j}\varphi \rangle + \zeta\langle \nabla_{e_j}\varphi, e_j \cdot \varphi \rangle + \zeta^2 \langle e_j \cdot \varphi, e_j \cdot \varphi \rangle \Big) \\ &= \sum_{j=1}^{n} \Big(\langle \nabla_{e_j}\varphi, \nabla_{e_j}\varphi \rangle - \zeta\langle \varphi, e_j \cdot \nabla_{e_j}\varphi \rangle - \zeta\langle e_j \cdot \nabla_{e_j}\varphi, \varphi \rangle + \zeta^2 \langle \varphi, \varphi \rangle \Big) \\ &= \langle \nabla\varphi, \nabla\varphi \rangle - \zeta\langle \varphi, D\varphi \rangle - \zeta\langle D\varphi, \varphi \rangle + n\zeta^2 \langle \varphi, \varphi \rangle . \end{aligned}$$

Integrating this yields

$$(\nabla^{-\zeta}\varphi, \nabla^{-\zeta}\varphi) = (\nabla\varphi, \nabla\varphi) - 2\zeta(D\varphi, \varphi) + n\zeta^2(\varphi, \varphi) . \tag{3.8}$$

We also have

$$(D - \zeta)^2\varphi = (D - \zeta)(D\varphi - \zeta\varphi) = D^2\varphi - 2\zeta D\varphi + \zeta^2 D\varphi .$$

Integrating and using the Lichnerowicz formula and Proposition 3.29 we obtain

$$\begin{aligned} ((D - \zeta)^2\varphi, \varphi) = (D^2\varphi - 2\zeta D\varphi + \zeta^2\varphi, \varphi) &= (\Delta\varphi, \varphi) + ((1/4\,\mathrm{scal} + \zeta^2)\varphi, \varphi) - 2\zeta(D\varphi, \varphi) \\ &= (\nabla\varphi, \nabla\varphi) + ((1/4\,\mathrm{scal} + \zeta^2)\varphi, \varphi) - 2\zeta(D\varphi, \varphi) . \end{aligned} \tag{3.9}$$

Let λ be an eigenvalue of D with corresponding eigenspinor $\varphi \in \Gamma(M, \Sigma M)$. Set $\zeta := \lambda/n$. From (3.8) we obtain

$$(\nabla^{-\lambda/n}\varphi, \nabla^{-\lambda/n}\varphi) = (\nabla\varphi, \nabla\varphi) - 2\frac{\lambda^2}{n}(\varphi, \varphi) + n\frac{\lambda^2}{n^2}(\varphi, \varphi) = (\nabla\varphi, \nabla\varphi) - \frac{\lambda^2}{n}(\varphi, \varphi) .$$

Combining this with (3.9) yields

$$\begin{aligned}\left(\lambda - \frac{\lambda}{n}\right)^2 (\varphi, \varphi) &= ((D - {}^{\lambda}\!/\!_{n})^2 \varphi, \varphi) = (\nabla\varphi, \nabla\varphi) + \left(\left(\frac{1}{4}\operatorname{scal} + \frac{\lambda^2}{n^2}\right)\varphi, \varphi\right) - 2\frac{\lambda^2}{n}(\varphi, \varphi) \\ &= (\nabla^{\lambda/n}\varphi, \nabla^{\lambda/n}\varphi) + \left(\frac{\lambda^2}{n^2} - \frac{\lambda^2}{n}\right)(\varphi, \varphi) + \frac{1}{4}(\operatorname{scal}\varphi, \varphi)\,.\end{aligned}$$

Substracting $\lambda^2 {}^{(1-n)}\!/\!_{n^2} (\varphi, \varphi)$ from both sides we obtain

$$\lambda^2 \frac{n-1}{n}(\varphi, \varphi) = (\nabla^{\lambda/n}\varphi, \nabla^{\lambda/n}\varphi) + \frac{1}{4}(\operatorname{scal}\varphi, \varphi) \geqslant \frac{\operatorname{scal}_0}{4}(\varphi, \varphi)\,, \tag{3.10}$$

which is the desired inequality.

Now assume that $\lambda = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}\operatorname{scal}_0}$. Then we have equality in (3.10), which implies $\nabla^{\lambda/n}\varphi = 0$, i.e., φ is a Killing spinor with Killing number ${}^{\lambda}\!/\!_{n} = \mp\frac{1}{2}\sqrt{\frac{1}{n(n-1)}\operatorname{scal}_0}$ and the scalar curvature is automatically constant. □

3.4. **Conformal Covariance and Twistors.** In this section we will show that the Dirac operator is conformally covariant, i.e., it transforms nicely w.r.t. a conformal change of the metric. Then, we introduce so called twistor spinors. The space of twistor spinors can be seen as a conformally invariant extension of the space of Killing spinors. The final result of this section will be that in the presence of twistor spinors, we can always conformally change the metric in such a way that the new metric is Einstein. Roger Penrose introduced twistor spinors in his work on general relativity, for which he was awarded the Nobel prize in physics in 2020.

Definition 3.65. *Let M be a smooth manifold. Two Riemannian metrics g and h on M are called conformal if there exists a function $u \in C^\infty(M)$ such that $h = e^{2u}g$.*

Remark 3.66. *A conformal change of the metric leads to a change of lengths of tangent vectors. Angles, on the other hand, are preserved.*

Proposition 3.67. *Let (M, g) be an oriented Riemannian manifold with a Spin-structure (P, π_g). Let $u \in C^\infty(M)$ and consider the conformal metric $h = e^{2u}g$. Define*

$$\psi_u := \psi : \mathrm{SO}(M, h) \ni v_x = (v_1, \dots, v_n) \mapsto (e^{u(x)}v_1, \dots, e^{u(x)}v_n) \in \mathrm{SO}(M, g)\,,$$

$Q := P$, $\pi_Q := \pi_P$, $\Psi_u := \Psi : Q \ni q \mapsto q \in P$ and $\pi_h := \psi^{-1} \circ \pi_g \circ \Psi$. Then, ψ is an $\mathrm{SO}(n)$-principal fibre bundle isomorphism, $(Q, \pi_Q; \mathrm{Spin}(n))$ is a $\mathrm{Spin}(n)$-principal fibre bundle over M, $\Psi : Q \to P$ is a $\mathrm{Spin}(n)$-principal fibre bundle isomorphism and (Q, π_h) is a Spin-structure on (M, h). The situation can be visualized by the commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\Psi_u"] \arrow[d, "\pi_h"] \arrow[ddr, bend right=60, "\pi_Q"'] & P \arrow[d, "\pi_g"] \arrow[dd, bend left=60, "\pi_P"] \\
\mathrm{SO}(M,h) \arrow[r, "\psi_u"] \arrow[dr] & \mathrm{SO}(M,g) \arrow[d] \\
& M
\end{tikzcd}$$

Proof. The map ψ is obviously a smooth bijection preserving fibres over M. Its $\mathrm{SO}(n)$-equivariance is a straightforward calculation.

As an abstract $\mathrm{Spin}(n)$-principal fibre bundle, $(Q, \pi_Q; \mathrm{Spin}(n))$ is just $(P, \pi_P; \mathrm{Spin}(n))$ and Ψ is the identity, so we only need to prove that it is a Spin-structure for (M, h). Let $q \in Q$ and $a \in \mathrm{Spin}(n)$. Then

$$\pi_h(q \cdot a) = \psi^{-1} \circ \pi_g \circ \Psi(q \cdot a) = \psi^{-1} \circ \pi_g(q \cdot a) = \psi^{-1}(\pi_g(q) \cdot \lambda(a)) = \psi^{-1}(\pi_g(q)) \cdot \lambda(a) = \pi_h(q) \cdot \lambda(a)\,.$$

□

Corollary 3.68. *In the situation of Proposition 3.67, denote the spinor bundles associated with the Spin-structures (P,π_g) over (M,g) and (Q,π_h) over (M,h) by $\Sigma_g M$ and $\Sigma_h M$, respectively. Then the principal fibre bundle isomorphism Ψ_u induces a vector bundle isomorphism $\Psi_u : \Sigma_h M \to \Sigma_g M$ which preserves bundle metrics and satisfies*

$$\Psi_u(v\cdot\sigma) = \mathrm{e}^{u(x)} v\cdot\Psi_u(\sigma)$$

for all $v \in T_xM$, $\sigma \in \Sigma_h M_x$, $x \in M$, where $\cdot$ on the left-hand side denotes Clifford multiplication in $\Sigma_h M$ and in $\Sigma_g M$ on the right-hand side.

Proof. Since the abstract $\mathrm{Spin}(n)$-bundles of the Spin-structures are identical, the induced isomorphism on the spinor bundles is simply given by $\Psi_u : \Sigma_h M \ni [p,\sigma] \mapsto [p,\sigma] \in \Sigma_g M$. The formula relating the Clifford multiplications is a straightforward consequence of the definition of Clifford multiplication (cmp. proof of Proposition 3.11). □

Remark 3.69. *The following formula relating the Levi-Civita connections ∇^g and ∇^h of the conformal Riemannian metrics g resp. h with $h = \mathrm{e}^{2u} g$ is well-known, see, e.g.,* [Be87]*:*

$$\nabla^h_X Y = \nabla^g_X Y + \mathrm{d}u(X)Y + \mathrm{d}u(Y)X - g(X,Y)\,\mathrm{grad}_g\, u \quad \textit{for all} \quad X,Y \in \mathcal{V}(M)\,.$$

Proposition 3.70. *In the situation of Proposition 3.67, denote the spinor connections in the spinor bundles $\Sigma_g M$ and $\Sigma_h M$ by ∇^g and ∇^h, respectively. Then we have*

$$\nabla^h_X = \Psi_u^{-1} \circ \left(\nabla^g_X - \frac{1}{2} X\cdot \mathrm{grad}_g\, u - \frac{1}{2}X(u)\right)\circ \Psi_u \quad \textit{for all} \quad X \in \mathcal{V}(M)\,.$$

Proof. Let $s : M \supseteq U \to P = Q$ be a local section and $\pi_g \circ s = (e_1,\dots,e_n) : U \to \mathrm{SO}(M,g)$, $\pi_h \circ s = (v_1,\dots,v_n) : U \to \mathrm{SO}(M,h)$ the associated loca g- resp. h-OONBs. Note that we have $v_j = \mathrm{e}^{-u} e_j$ for all $j = 1,\dots,n$. Let $w \in C^\infty(U,\Sigma_n)$ and $\varphi = [s,w] \in \Gamma(U,\Sigma_h M)$. Then for every $X \in \Gamma(U,TM)$ we have

$$\begin{aligned}
\nabla^h_X\varphi &= [s,X(w)] + \frac{1}{4}\sum_{j=1}^n v_j\cdot\nabla^h_X v_j\cdot\varphi = [s,X(w)] + \frac{1}{4}\mathrm{e}^{-u}\sum_{j=1}^n e_j\cdot\nabla^h_X(\mathrm{e}^{-u}e_j)\cdot\varphi\\
&= [s,X(w)] + \frac{1}{4}\mathrm{e}^{-u}\sum_{j=1}^n e_j\cdot\left(-X(u)\mathrm{e}^{-u}e_j + \mathrm{e}^{-u}\nabla^h_X e_j\right)\cdot\varphi = [s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\left(nX(u) + \sum_{j=1}^n e_j\cdot\nabla^h_X e_j\right)\cdot\varphi\\
&= [s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\left(nX(u) + \sum_{j=1}^n e_j\cdot\left(\nabla^g_X e_j + \mathrm{d}u(X)e_j + \mathrm{d}u(e_j)X - g(X,e_j)\,\mathrm{grad}_g\, u\right)\right)\cdot\varphi\\
&= [s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\left(nX(u) + \sum_{j=1}^n e_j\cdot\nabla^g_X e_j - nX(u) + \mathrm{grad}_g\, u\cdot X - X\cdot\mathrm{grad}_g\, u\right)\cdot\varphi\\
&= [s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\left(\sum_{j=1}^n e_j\cdot\nabla^g_X e_j - 2X\cdot\mathrm{grad}_g\, u - 2h(\mathrm{grad}_g\, u, X)\right)\cdot\varphi\,,
\end{aligned}$$

so that, using the formula in the above corollary, we obtain

$$\begin{aligned}
\nabla^h_X\varphi &= \Psi_u^{-1}\circ\Psi_u\left([s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\left(\sum_{j=1}^n e_j\cdot\nabla^g_X e_j - 2X\cdot\mathrm{grad}_g\, u - 2h(\mathrm{grad}_g\, u,X)\right)\cdot\varphi\right)\\
&= \Psi_u^{-1}\left([s,X(w)] + \frac{1}{4}\mathrm{e}^{-2u}\mathrm{e}^{2u}\sum_{j=1}^n e_j\cdot\nabla^g_X e_j - \frac{1}{2}\mathrm{e}^{-2u}\mathrm{e}^{2u}X\cdot\mathrm{grad}_g\, u - \frac{1}{2}\mathrm{e}^{-2u}h(\mathrm{grad}_g\, u,X)\right)\Psi_u\circ\varphi\\
&= \Psi_u^{-1}\left(\nabla^g_X - \frac{1}{2}X\cdot\mathrm{grad}_g\, u - \frac{1}{2}X(u)\right)\Psi_u\circ\varphi\,.
\end{aligned}$$

□

Corollary 3.71. *Denoting the Dirac operators acting on sections of $\Sigma_g M$ and $\Sigma_h M$ by D_g and D_h, respectively, we have*

$$D_h = \mathrm{e}^{-\frac{n+1}{2}u}\Psi_u^{-1}\circ D_g\circ\left(\mathrm{e}^{\frac{n-1}{2}u}\Psi_u\right)\,.$$

Proof. By the last proposition, we have

$$\begin{aligned}
D_h &= \sum_{j=1}^n v_j \cdot \nabla^h_{v_j} = \sum_{j=1}^n v_j \cdot \Psi_u^{-1}\left(\nabla^g_{v_j} - \frac{1}{2} v_j \cdot \operatorname{grad}_g u - \frac{1}{2} v_j(u)\right) \circ \Psi_u \\
&= \mathrm{e}^u \Psi_u^{-1}\left(\sum_{j=1}^n v_j \cdot \nabla^g_{v_j} - \frac{1}{2}\sum_{j=1}^n v_j \cdot v_j \cdot \operatorname{grad}_g u - \frac{1}{2}\sum_{j=1}^n v_j \cdot v_j(u)\right) \circ \Psi_u \\
&= \mathrm{e}^u \Psi_u^{-1}\left(\mathrm{e}^{-2u}\sum_{j=1}^n e_j \cdot \nabla^g_{e_j} + \mathrm{e}^{-2u}\frac{n}{2} \cdot \operatorname{grad}_g u - \mathrm{e}^{-2u}\frac{1}{2}\operatorname{grad}_g u\right) \circ \Psi_u \\
&= \mathrm{e}^{-u}\Psi_u^{-1}\left(D_g + \frac{n-1}{2}\operatorname{grad}_g u\right) \circ \Psi_u\,.
\end{aligned}$$

Note that we have for every $\alpha \in \mathbb{R}$,

$$D_g(\mathrm{e}^{\alpha u}\varphi) = \operatorname{grad} \mathrm{e}^{\alpha u} \cdot \varphi + \mathrm{e}^{\alpha u} D_g \varphi = \mathrm{e}^{\alpha u}\left(\alpha \operatorname{grad} u \cdot \varphi + D_g \varphi\right)\,.$$

Setting $\alpha = \frac{n-1}{2}$ yields the claimed formula. □

Lemma 3.72. *Let (M, g) be a Riemannian spin manifold with a fixed Spin-structure. Consider the vector bundle*

$$\ker \mu := \bigcup_{x \in M} \ker \mu_x \subseteq TM \otimes \Sigma M\,.$$

Then

$$\begin{aligned}
P : TM \otimes \Sigma M &\to TM \otimes \Sigma M \\
v \otimes \sigma &\mapsto v \otimes \sigma + \frac{1}{n}\sum_{j=1}^n e_j \otimes e_j \cdot v \cdot \sigma\,,
\end{aligned}$$

where $(e_1, \ldots, e_n)$ is an ONB of the appropriate tangent space, is an orthogonal projection with image $\operatorname{im} P = \ker \mu$.

Proof. Exercise. □

Definition 3.73. *Let (M, g) be a Riemannian spin manifold with a fixed Spin-structure. We call $\mathcal{T} := P \circ (\sharp \otimes \mathrm{id}_{\Sigma M}) \circ \nabla : \Gamma(M; \Sigma M) \to \Gamma(M; TM \otimes \Sigma M)$ the twistor operator . Any spinor $\varphi \in \Gamma(M; \Sigma M)$ with $\varphi \in \ker \mathcal{T}$ is called a twistor spinor or simply twistor .*

Lemma 3.74. *We have $\varphi \in \ker \mathcal{T}$ if and only if*

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \quad \textit{for all} \quad X \in \mathcal{V}(M)\,.$$

Proof. Let $(e_1, \ldots, e_n)$ be a local ONB with g-dual ONB $(\varepsilon_1, \ldots, \varepsilon_n)$. Then

$$\sharp \otimes \mathrm{id}_{\Sigma M} \circ \nabla \varphi = \sharp \otimes \mathrm{id}_{\Sigma M}\left(\sum_{j=1}^n \varepsilon_j \otimes \nabla_{e_j}\varphi\right) = \sum_{j=1}^n e_j \otimes \nabla_{e_j}\varphi$$

so that

$$\begin{aligned}
\mathcal{T}(\varphi) &= \sum_{j=1}^n e_j \otimes \nabla_{e_j}\varphi + \frac{1}{n}\sum_{i,j=1}^n e_i \otimes e_i \cdot e_j \cdot \nabla_{e_j}\varphi = \sum_{j=1}^n e_j \otimes \nabla_{e_j}\varphi + \frac{1}{n}\sum_{j=1}^n e_j \otimes e_j \cdot D\varphi \\
&= \sum_{j=1}^n e_j \otimes \left(\nabla_{e_j}\varphi + \frac{1}{n} e_j \cdot D\varphi\right) = 0
\end{aligned}$$

if and only if

$$\nabla_{e_j}\varphi + \frac{1}{n} e_j \cdot D\varphi = 0 \quad \text{for all} \quad j = 1, \ldots, n\,,$$

which is obviously equivalent to the claim of the lemma. □

Proposition 3.75. *In the situation of Proposition 3.67, denote by $\mathcal{T}_g$ and $\mathcal{T}_h$ the twistor operators associated with the spinor bundles $\Sigma_g M$ and $\Sigma_h M$, respectively. Then we have*

$$\mathcal{T}_h(\varphi) = \mathrm{id}_{TM} \otimes \Psi_u^{-1}(\mathrm{e}^{-\frac{u}{2}} \mathcal{T}_g(\mathrm{e}^{-\frac{u}{2}} \Psi_u \circ \varphi)) \quad \text{for all} \quad \varphi \in \Gamma(M; \Sigma_h M)\,.$$

In particular, $\varphi \in \Gamma(M;\Sigma_h M)$ is a twistor spinor if and only if $\mathrm{e}^{-\frac{u}{2}}\Psi_u \circ \varphi \in \Gamma(M;\Sigma_g M)$ is a twistor spinor.

Proof. Straightforward computation using Proposition 3.70, Corollary 3.71 and Lemma 3.74. □

Proposition 3.76. *Let (M,g) be a Riemannian spin manifold with a fixed Spin-structure. If $\varphi \in \Gamma(M;\Sigma M)$ is a twistor, then*

$$\nabla_X D\varphi = \frac{n}{2(n-2)} \left(\frac{\mathrm{scal}}{2(n-1)} X - \mathrm{ric}(X) \right) \cdot \varphi \quad \text{for all} \quad X \in \mathcal{V}(M)\,.$$

Proof. Exercise. □

Proposition 3.77. *Let (M,g) be a Riemannian spin manifold with a fixed Spin-structure. Let $K : TM \ni X \mapsto \frac{1}{n-2}\left(\frac{\mathrm{scal}}{2(n-1)} X - \mathrm{ric}(X)\right) \in TM$. Consider the complex vector bundle $S := \Sigma M \oplus \Sigma M$ over M with the covariant derivative*

$$\nabla^S_X := \begin{pmatrix} \nabla_X & \frac{1}{n} X \\ -\frac{n}{2} K(X) & \nabla_X \end{pmatrix}\,.$$

Then, for any twistor $\varphi \in \Gamma(M;\Sigma M)$ we have $\nabla^S \binom{\varphi}{D\varphi} = 0$. Conversely, if $\binom{\varphi}{\psi} \in \Gamma(M;S)$ satisfies $\nabla^S\binom{\varphi}{\psi} = 0$, then φ is a twistor and $\psi = D\varphi$.

Proof. If $\varphi \in \Gamma(M;\Sigma M)$ is a twistor, then Lemma 3.74 and the last proposition imply that $\nabla^S\binom{\varphi}{D\varphi} = 0$.

Now let $\binom{\varphi}{\psi} \in \Gamma(M;S)$ be ∇^S-parallel. By definition of ∇^S, we have

$$\nabla_X \varphi + \frac{1}{n} X \cdot \psi = 0$$

for all $X \in \mathcal{V}(M)$. Multiplying this equation by X we obtain

$$X \nabla_X \varphi - \frac{1}{n}\|X\|^2 \psi = 0\,.$$

Choosing $X = e_1, \ldots, e_n$ for a local ONB and summing the resulting equations yields

$$D\varphi - \psi = 0\,,$$

which also shows that φ is a twistor. □

Remark 3.78. *By the last proposition, twistor spinors are in 1:1-correspondence with ∇^S-parallel sections of the bundle S. Since parallel sections on a connected manifold are uniquely determined by their value at one point, we conclude that a twistor spinor φ is uniquely determined by $(\varphi(p), D\varphi(p))$ for an arbitrary $p \in M$ and that the twistor space has dimension at most $2 \cdot 2^{\lfloor \frac{n}{2} \rfloor}$.*

Theorem 3.79. *Let (M,g) be a connected Riemannian spin manifold with a fixed Spin-structure. If $\varphi \in \Gamma(M;\Sigma M)$ is a nontrivial twistor, then $\mathrm{null}\varphi := \{p \in M \mid \varphi(p) = 0\}$ is a discrete set.*

Proof. Denote $\ell_\varphi := \langle \varphi, \varphi \rangle$, the squared length function of the twistor φ and let $p \in \mathrm{null}\varphi$. Let $X, Y \in \mathcal{V}(M)$ be any two vector fields. Firstly, we have

$$X(\ell_\varphi)(p) = \langle \nabla_X \varphi, \varphi \rangle_p + \langle \varphi, \nabla_X \varphi \rangle_p = 0\,,$$

so that p is a critical point of ℓ_φ.

Secondly,

$$\begin{aligned} Y(X(\ell_\varphi))(p) &= 2(Y\Re\langle \nabla_X\varphi, \varphi\rangle)_p = -\frac{2}{n} Y \Re\langle X \cdot D\varphi, \varphi\rangle_p = -\frac{2}{n}\left(\Re\langle \nabla_Y(X\cdot D\varphi), \varphi\rangle_p + \Re\langle X \cdot D\varphi, \nabla_Y \varphi\rangle_p\right) \\ &= \frac{2}{n^2}\Re\langle X \cdot D\varphi, Y \cdot D\varphi\rangle_p = \frac{2}{n^2} g(X,Y)_p \|D\varphi\|_p^2\,. \end{aligned}$$

Hence, the hessian of ℓ_φ at p is given by

$$\mathrm{Hess}_p\, \ell_\varphi(X,Y) = \nabla(\mathrm{d}\ell_\varphi)(X,Y)_p = \nabla_X(\mathrm{d}\ell_\varphi)(Y)_p = X(Y(\ell_\varphi))_p - \mathrm{d}(\ell_\varphi)_p(\nabla_X Y_p) = \frac{2}{n^2} g(X,Y)_p \|D\varphi\|_p^2\,.$$

In case $D\varphi(p) \neq 0$, p is a nondegenerate critical point ℓ_φ and thus an isolated zero point. In case $D\varphi(p) = 0$ we have $\varphi \equiv 0$ by the last remark, which contradicts the assumption of the theorem. □

Lemma 3.80. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure. If $\varphi \in \Gamma(M; \Sigma M)$ is a twistor, the functions*

$$C_\varphi := \Re\langle D\varphi, \varphi\rangle \,,$$
$$Q_\varphi := \|\varphi\|^2\|D\varphi\|^2 - C_\varphi^2 - \sum_{j=1}^n (\Re\langle D\varphi, e_j \cdot \varphi\rangle)^2 \,,$$

where $(e_1, \dots, e_n)$ is an ONB, are constant.

Proof. Exercise. □

Theorem 3.81. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure. If $\varphi \in \Gamma(M; \Sigma M)$ is a twistor with $\|\varphi\|^2 = 1$, then (M, g) is an Einstein manifold with scalar curvature*

$$\mathrm{scal} = \frac{n}{4(n-1)}(C_\varphi^2 + Q_\varphi) \,.$$

Proof. Since $\ell_\varphi \varphi = 1$, we have

$$0 = X\ell_\varphi = 2\Re\langle \nabla_X \varphi, \varphi\rangle = -\frac{2}{n}\Re\langle X \cdot D\varphi, \varphi\rangle = \frac{2}{n}\Re\langle D\varphi, X \cdot \varphi\rangle \,.$$

This automatically implies by Proposition 3.76 that

$$X\langle D\varphi, D\varphi\rangle = 2\Re\langle \nabla_X D\varphi, D\varphi\rangle = n\Re\langle K(X) \cdot \varphi, D\varphi\rangle = 0 \,,$$

that is, $\|D\varphi\|^2$ is constant.

Now, for any vector fields X, Y we have

$$\begin{aligned}\frac{n}{2}g(K(X), Y) &= \frac{n}{2}g(K(X), Y)\|\varphi\|^2 = \frac{n}{2}\Re\langle K(X) \cdot \varphi, Y \cdot \varphi\rangle = \Re\langle \nabla_X D\varphi, Y \cdot \varphi\rangle \\ &= X\Re\langle D\varphi, Y \cdot \varphi\rangle - \Re\langle D\varphi, \nabla_X (Y \cdot \varphi)\rangle \\ &= -\Re\langle D\varphi, \nabla_X Y \cdot \varphi\rangle - \Re\langle D\varphi, Y \cdot \nabla_Y \varphi\rangle \\ &= \frac{1}{n}\Re\langle D\varphi, Y \cdot X \cdot D\varphi\rangle = -\frac{1}{n}g(X, Y)\|D\varphi\|^2 \,,\end{aligned}$$

from which we conclude that $K(X) = \frac{1}{n-2}\left(\frac{\mathrm{scal}}{2(n-1)}X - \mathrm{ric}(X)\right)$ is a constant multiple of the identity, i.e., (M, g) is an Einstein manifold.

Regarding the sectional curvature, we compute

$$\begin{aligned}C_\varphi^2 + Q_\varphi &= \|\varphi\|^2\|D\varphi\|^2 - \sum_{j=1}^n (\Re\langle D\varphi, e_j \cdot \varphi\rangle)^2 = 1 \cdot \|D\varphi\|^2 = -\frac{n^2}{2}\frac{g(K(X), X)}{g(X, X)} \\ &= -\frac{n^2}{2}\frac{1}{n-2}\left(\frac{\mathrm{scal}}{2(n-1)} - \frac{\mathrm{scal}}{n}\right) = \frac{n}{4(n-1)}\,\mathrm{scal} \,,\end{aligned}$$

where we have used that for an Einstein manifold with $\mathrm{Ric} = \lambda g$ we have $\mathrm{scal} = n\lambda$. □

Corollary 3.82. *Let (M, g) be a connected Riemannian spin manifold with a fixed Spin-structure. Assume there exists a nontrivial twistor $\varphi \in \Gamma(M; \Sigma M)$. On $N := M \backslash \mathrm{null}\varphi$, set $h := \frac{1}{\|\varphi\|^4}g$. Then (N, h) is an Einstein manifold with nonnegative scalar curvature*

$$\mathrm{scal}_h = \frac{n}{4(n-1)}(C_\varphi^2 + Q_\varphi) \,.$$

Proof. Since $\mathrm{null}\varphi$ is a discrete set, $M \backslash \mathrm{null}\varphi$ is indeed a manifold. The twistor φ is obviously also a twistor on $(N, g_{|N})$. By Proposition 3.75, with $\mathrm{e}^{2u} = 1/\|\varphi\|^4$, we have that $\mathrm{e}^{u/2}\Psi_u^{-1} \circ \varphi \in \Gamma(N, \Sigma_h N)$ is a twistor with norm

$$\|\mathrm{e}^{u/2}\Psi_u^{-1} \circ \varphi\|^2 = \mathrm{e}^u\|\Psi_u^{-1} \circ \varphi\|^2 = \frac{1}{\|\varphi\|^2}\|\varphi\|^2 = 1 \,,$$

so that the claim follows from the last Theorem. □

Remark 3.83. *One can show that if $\frac{4n}{n-1}(C_\varphi^2 + Q_\varphi) > 0$ then $1/\|\varphi\|\Psi_u^{-1} \circ \varphi$ is the sum of two real killing spinors whereas if $\frac{4n}{n-1}(C_\varphi^2 + Q_\varphi) = 0$, then $1/\|\varphi\|\Psi_u^{-1} \circ \varphi$ is parallel.*

4. Outlook: $\mathrm{Spin}^{\mathbb{C}}$- and generalized Dirac operators

In this course, we have treated the Spin-Dirac operator, also known as the fundamental or Atiyah-Singer-Dirac operator. While this is undoubtedly the most important Dirac operator, it is by far not the only one.

First we want to discuss the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator which is motivated by physics. As a starting point we take Theorem 3.13 which asserted that there is a unique spinor connection ∇ on ΣM satisfying the Leibniz rule (3.1) w.r.t. Clifford multiplication. While uniqueness is often times a desirable property, it hinders us if we would like to introduce a magnetic field into the picture, i.e., a certain imaginary valued object. To do this, we go back to the beginning and consider the spin group and its complex fundamental representation, which was given by the restriction of an irreducible representation of $\mathbb{C}\ell_n$ to $\mathrm{Spin}(n)$,

$$\mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n \subseteq \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)\,.$$

The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is the subgroup of $\mathbb{C}\ell_n^*$ generated by $\mathrm{Spin}(n)$ and S^1. Since $\mathrm{Spin}(n) \cap S^1 = \{\pm 1\}$ we can identify $\mathrm{Spin}^{\mathbb{C}}(n)$ with $\mathrm{Spin}(n) \times_{\mathbb{Z}^2} S^1 = \mathrm{Spin}(n) \times S^1/\sim$, where $[g,z] \sim [-g,-z]$. The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is now a double cover for $\mathrm{SO}(n) \times \mathrm{U}(1)$ ($\mathrm{U}(1) \simeq S^1$), where the covering map is given by

$$\lambda \times \ell : \mathrm{Spin}^{\mathbb{C}}(n) \ni [g,z] \mapsto (\lambda(g), \ell(z)) = (\lambda(g), z^2) \in \mathrm{SO}(n) \times \mathrm{U}(1)\,.$$

We again define the fundamental representation of $\mathrm{Spin}^{\mathbb{C}}(n)$ by the restriction of an irreducible representation of $\mathbb{C}\ell_n$,

$$\kappa_n : \mathrm{Spin}^{\mathbb{C}}(n) \subseteq \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)\,,$$

and the statement in Propopsition 1.49 holds verbatim in this case too.

Next we want to define $\mathrm{Spin}^{\mathbb{C}}$-structures on an oriented Riemannian manifold (M,g). To do this, we need an additional datum, a $\mathrm{U}(1)$-principal fibre bundle Q over M. This object is in general not canonical, i.e., this represents a degree of freedom.

A $\mathrm{Spin}^{\mathbb{C}}$-structure on an oriented Riemannian manifold (M,g) is a pair (P,π) consiting of a $\mathrm{Spin}^{\mathbb{C}}(n)$-principal fibre bundle P over M and a $\lambda \times \ell$-equivariant map $\pi : P \to \mathrm{SO}(M,g) \times Q$, that is, the following diagram is commutative,

$$\begin{tikzcd}
P \times \mathrm{Spin}^{\mathbb{C}}(n) \arrow[r, "\cdot"] \arrow[d, "\pi\times\lambda\times\ell"] & P \arrow[d, "\pi"] \arrow[dr, "\pi_P"] & \\
\mathrm{SO}(M,g) \times Q \times \mathrm{SO}(n) \times \mathrm{U}(1) \arrow[r, "\cdot"] & \mathrm{SO}(M) \times Q \arrow[r, "\pi_{\mathrm{SO}(n)} \times \pi_Q"'] & M
\end{tikzcd}$$

Similarly to the spin case, the existence of $\mathrm{Spin}^{\mathbb{C}}$-structures does not depend on the geometry of the manifold, but only on its topology.

The spinor bundle is now defined as in the spin case,

$$\Sigma M := P \times_{\kappa_n} \Sigma_n\,.$$

We also define Clifford multiplication analogously to the spin case and all statements made there hold in the the $Spin^{\mathbb{C}}$ case too.

In order to define the Dirac operator, we need a spinor connection on ΣM. Contrary to the spin situation, there is no unique lift of the Levi-Civita ∇^{LC} connection to the spin bundle. Rather, we have to choose a connection ∇^L in the complex line bundle $L := Q \times_{\rho_1} \mathbb{C}$ first (here, $\rho_1 : \mathrm{U}(1) \to \mathrm{Gl}(\mathbb{C})$ is the standard representation of $\mathrm{U}(1)$, and L is the complex vector bundle associated with Q and ρ_1). Assocaited to the pair $(\nabla^{\mathrm{LC}}, \nabla^L)$ is now a unique connection ∇ in the spinor bundle ΣM, which is metric and satisfies (3.1). The $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator D is defined analogously to the usual Dirac operator, i.e., as the superposition of the spinor connection and Clifford multiplication.

For many of the theorems that we have seen in this course there are analogues for the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator. For example, the Lichnerowicz formula reads

$$D^2 = \Delta + \frac{1}{4}\,\mathrm{scal} + \frac{1}{2}\Omega\,.$$

Here, Ω is the curvature of the line bundle L, which, since L is 1-dimensional, is an alternating tensor $\Omega : TM \oplus TM \to \mathrm{i}\mathbb{R}$, and Clifford multiplication can be extended to alternating tensors.

One point that makes the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator attractive is that the class of $\mathrm{Spin}^{\mathbb{C}}$ manifolds is much larger than the class of spin manifolds and that it contains the latter. More precisely,

- every spin manifold (M, g) with a fixed Spin-structure P has an associated canonical $\mathrm{Spin}^{\mathbb{C}}$-structure whose Dirac operator can be canonically and isometrically identified with the Spin-Dirac operator we started with.
- Every (almost) complex manifold carries a canonical $\mathrm{Spin}^{\mathbb{C}}$-structure.

The last subject we want to touch briefly is that of generalized Dirac operators. The question we ask ourselves is: Can we abstract the Spin-Dirac operator to create a framework for operators which behave like the Spin-Dirac operator? One way to do this is via so-called Dirac bundles (see [LM89]; but note that there are even more general notions of Dirac operators). Given a Riemannian manifold (M, g), a Dirac bundle is a triple (S, μ, ∇^S) where

- S is a vector bundle over M equipped with a bundle metric,
- $\mu : TM \otimes S \ni X \otimes \sigma \mapsto \mu(X \otimes \sigma) =: X \cdot \sigma \in S$ is a vector bundle homomorphism satisfying the Clifford relations
$$X \cdot (Y \cdot \sigma) + Y \cdot (X \cdot \sigma) = -2g(X, Y)\sigma \quad \text{for all} \quad X, Y \in T_xM, \sigma \in S_x, x \in M\,,$$
and which is orthogonal w.r.t. the bundle metric of S, i.e.,
$$\langle X \cdot \sigma, X \cdot \tau \rangle = \langle \sigma, \tau \rangle$$
for all $X \in T_xM$ with $\|X\| = 1$, $\sigma \in S_x$, $x \in M$,
- ∇^S is a metric covariant derivative in S satisfying
$$\nabla^S_X (Y \cdot \varphi) = \nabla^{\mathrm{LC}}_X Y \cdot \varphi + Y \cdot \nabla^S_X \varphi \quad \text{for all } X, Y \in \mathcal{V}(M), \varphi \in \Gamma(M, S)\,.$$

Given a Dirac bundle (S, μ, ∇^S) over a Riemannian manifold (M, g), the associated (generalized) Dirac operator is defined as

$$D : \Gamma(M, S) \xrightarrow{\nabla^S} \Gamma(M, T^*M \otimes S) \xrightarrow{\sharp \otimes \mathrm{id}} \Gamma(M, TM \otimes S) \xrightarrow{\mu} \Gamma(M, S)\,,$$

and is given locally by the familiar formula

$$D\varphi = \sum_{j=1}^{n} e_j \cdot \nabla^S_{e_j} \varphi\,.$$

Having made this definition, the question is whether there are any generalized Dirac operators. Certainly, every Spin-Dirac and $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator is a generalized Dirac operator. But are there any others?

One example is given by twisted Dirac operators. Suppose you are given a Riemannian $\mathrm{Spin}^{(\mathbb{C})}$ manifold (M, g) with fixed $\mathrm{Spin}^{(\mathbb{C})}$-structure. Take any vector bundle E over M and equip it with a bundle metric and a metric connection ∇^E. We define

- $S := \Sigma M \otimes E$,
- Clifford multiplication μ^S on S is just Clifford multiplication on the first factor, i.e.,
$$X \cdot (\sigma \otimes \tau) := (X \cdot \sigma) \otimes \tau\,,$$
- ∇^S as the canonical tensor product connection, i.e.,
$$\nabla^S_X(\varphi \otimes \psi) := \nabla^{\Sigma M}_X \varphi \otimes \psi + \varphi \otimes \nabla^E_X \psi\,.$$

Routine calculations show that (S, μ^S, ∇^S) is a Dirac bundle. The associated generalized Dirac operator is called the twisted Dirac operator with coefficients in E.

Generalized Dirac operators enjoy many of properties of the Spin-Dirac operator that we proved in this course, e.g.,

- $D(f\varphi) = \operatorname{grad} f \cdot \varphi + fD(\varphi)$ for all $f \in C^\infty(M)$, $\varphi \in \Gamma(M, S)$,
- Lichnerowicz fomulue
$$D^2 = \Delta^S + \mathscr{R}$$
where $\mathscr{R}$ is a certain function given in terms of the curvature tensor of S.

Often times, by choosing the right Dirac bundle, it is possible to connect the geometry with the topology of the underlying manifold through the associated Lichnerowicz formula and index theory (see [LM89]).

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